Geometry and Topology Graduate Exam
Spring 2014

Solve all **SEVEN** *problems. Partial credit will be given to partial solutions.*

Problem 1. Let X_n denotes the complement of n distinct points in the plane $\mathbb{R}^2$. Does there exist a covering map $X_2 \to X_1$? Explain.

Problem 2. Let $D = \{z \in \mathbb{C}; |z| \leq 1\}$ denote the unit disk, and choose a base point z_0 in the boundary $S^1 = \partial D = \{z \in \mathbb{C}; |z| = 1\}$. Let X be the space obtained from the union of D and $S^1 \times S^1$ by gluing each $z \in S^1 \subset D$ to the point $(z, z_0) \in S^1 \times S^1$. Compute all homology groups $H_k(X; \mathbb{Z})$.

Problem 3. Let $B^n = \{x \in \mathbb{R}^n; \|x\| \leq 1\}$ denote the n-dimensional closed unit ball, with boundary $S^{n-1} = \{x \in \mathbb{R}^n; \|x\| = 1\}$. Let $f\colon B^n \to \mathbb{R}^n$ be a continuous map such that $f(x) = x$ for every $x \in S^{n-1}$. Show that the origin 0 is contained in the image $f(B^n)$. (Hint: otherwise, consider $S^{n-1} \to B^n \xrightarrow{f} \mathbb{R}^n - \{0\}$.)

Problem 4. Consider the following vector fields defined in $\mathbb{R}^2$:

$$\mathbf{X} = 2\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \qquad \text{and} \qquad \mathbf{Y} = \frac{\partial}{\partial y}.$$

Determine whether or not there exists a (locally defined) coordinate system (s, t) in a neighborhood of $(x, y) = (0, 1)$ such that

$$\mathbf{X} = \frac{\partial}{\partial s}, \qquad \text{and} \qquad \mathbf{Y} = \frac{\partial}{\partial t}.$$

Problem 5. Let M be a differentiable (not necessarily orientable) manifold. Show that its cotangent bundle

$$T^*M = \{(x, u); x \in M \text{ and } u\colon T_xM \to \mathbb{R} \text{ linear}\}$$

is a manifold, and is orientable.

Problem 6. Calculate the integral $\int_{S^2} \omega$ where S^2 is the standard unit sphere in $\mathbb{R}^3$ and where ω is the restriction of the differential 2-form

$$(x^2 + y^2 + z^2)(x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy)$$

Problem 7. Let M be a compact m-dimensional submanifold of $\mathbb{R}^m \times \mathbb{R}^n$. Show that the space of points $x \in \mathbb{R}^m$ such that $M \cap \mathbb{R}^n$ is infinite has measure 0 in $\mathbb{R}^m$.

Geometry Spring 2014 (Incomplete)

① X_n is the complement of n distinct points in the plane $\mathbb{R}^2$. Is there a covering map $X_2 \to X_1$?

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solution

$X_1 \xrightarrow{p_n} X_1$ is a covering map, where $p_n(z) = z^n$. $\pi_1(X_1; x_0) \cong \mathbb{Z}$.

The image of $(p_n)_* : \mathbb{Z} \to \mathbb{Z}$ is $n\mathbb{Z}$.

Since X_1 is path connected, locally path connected, and semilocally simply connected, every path connected covering $\tilde{X}$ of X_1 is determined by the image of $\pi_1(\tilde{X}; \tilde{x}_0)$ in $\pi_1(X_1; x_0)$ (even w/out base points, since $\pi_1(X_1; x_0)$ is abelian). We've constructed covering spaces for every subgroup $n\mathbb{Z}$ of $\mathbb{Z}$, so we know therefore that every covering space of X_1 is either simply connected or homeomorphic to X_1. Since X_1 is not

homeomorphic to X_2 (the latter's fundamental group is the free group on two generators) and X_2 is not simply connected, it cannot be a covering space of X_1 (nor can X_n, $n>1$).
[Note in fact the universal cover of $\mathbb{R}^2 - \{0\}$ is $\mathbb{R}^2$.) $\square$.

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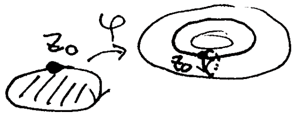
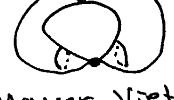
(2) $D = \{ z \in \mathbb{C} : |z| \leq 1 \}$ and choose a base point z_0 in the boundary $S^1 = \partial D = \{ z \in \mathbb{C} : |z| = 1 \}$.
Let X be the space obtained as $(D \amalg (S^1 \times S^1))/\sim$ where $\sim$ is determined by the gluing map $S^1 = \partial D \xrightarrow{\varphi} S^1 \times S^1$ where $\varphi(z) = (z, z_0)$. Compute the homology groups $H_k(X; \mathbb{Z})$.

<u>solution</u>



$\underset{\text{homotopy equivalent}}{\simeq}$



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$X_1 =$  $\simeq \{*\}$

X_2

$\shortparallel$

$\shortparallel$

$I \times S^1$.
$I^2 S^1$

$$\cdots \to H_k(S^1 \amalg S^1) \to H_k(\{*\}) \oplus H_k(S^1) \to H_k(X) \to H_{k-1}(S^1 \amalg S^1) \to$$

$k > 2$ / $\to 0 \to 0 \to H_k(X) \to 0 \to$

$$\boxed{H_k(X) \cong 0}$$

$k = 2$ / $\to 0 \to 0 \to H_2(X) \xrightarrow{f} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{g} \mathbb{Z} \to$

$(x,y) \mapsto x+y$

$$H_2(X) \cong \operatorname{im} f \cong \ker g = \{(x,y) : x+y=0\} = \{(x,-x)\} \cong \mathbb{Z}.$$

$$\boxed{H_2(X) \cong \mathbb{Z}}$$

$(x,y) \mapsto x+y$

$k = 1$ / $\to \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{f} \mathbb{Z} \xrightarrow{g} H_1(X) \xrightarrow{h} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\ell} \mathbb{Z} \oplus \mathbb{Z} \to$

$(x,y) \mapsto (-x-y, x+y)$

$\boxed{H_1(X) \cong \mathbb{Z}}$ $\boxed{H_0(X) \cong \mathbb{Z}}$

$\ker g \cong \operatorname{im} f = \mathbb{Z} \Rightarrow g = 0.$

$H_1(X) \cong \operatorname{im} h \cong \ker \ell = \{(x,y) : x+y=0\} = \{(x,-x)\} \cong \mathbb{Z}.$ $\square$.

③ Let $B^n = \{ X \in \mathbb{R}^n : \|X\| \leq 1 \}$, let $S^{n-1} = \partial B^n = \{ X \in \mathbb{R}^n : \|X\| = 1 \}$. Let $f : B \to \mathbb{R}^n$ be continuous s.t. $f(X) = X$ for all $X \in S^{n-1}$. Show the origin is contained in the image $f(B^n)$. We Assume $n \geq 2$.

<u>Solution</u> Suppose otherwise.

Consider
$$\begin{tikzcd} S^{n-1} \arrow[r, "i"] & B^n \arrow[r, "f"] & \mathbb{R}^n - \{0\} \arrow[d, "\pi", "\wr"'] \\ & & S^{n-1} \end{tikzcd}.$$

So this induces a map

$$\begin{tikzcd} H_{n-1}(S^{n-1}) \arrow[r] \arrow[rr, bend right, "0"'] & H_{n-1}(\underset{0}{B^n}) \arrow[r] & H_{n-1}(S^{n-1}) \end{tikzcd}.$$

However, $\pi \circ f \circ i$ equals the identity $S^{n-1} \to S^{n-1}$, so the map $H_{n-1}(S^{n-1}) \to H_{n-1}(S^{n-1})$ is also the identity map, in addition to being the zero map. This contradicts $H_{n-1}(S^{n-1}) \cong \mathbb{Z}$. $\square$.

(4) Vector fields in $\mathbb{R}^2$:

$$X = 2\partial_x + x\,\partial_y,$$
$$Y = \partial_y.$$

Determine whether ~~there~~ is a local coordinate system (s,t) around $(x,y)=(0,1)$ s.t. $X = \partial_s$, $Y = \partial_t$.

<u>Solution</u> It is a theorem that (from Bonahon's HW) if $[X,Y]=0$, then there is such a local coordinate system around every point. Recall $[\partial_i,\partial_j]=0$ and $[fX,gY]=f(Xg)Y-g(Yf)X + fg[X,Y]$.

$$[X,Y] = 2[\partial_x,\partial_y] + [x\partial_y,\partial_y] =$$
$$= 0 + x\,\cancelto{0}{\partial_y(1)}\,\partial_y - \cancelto{0}{\partial_y(x)}\,\partial_y + x\,\cancelto{0}{[\partial_y,\partial_y]} = 0. \quad \square$$

(5) (augmented) M smooth mfld, Not necessarily oriented. (a) Show the tangent bundle TM is a manifold that is orientable.

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solution to (a) Let $\{U_i, \varphi_i\}_{i\in I}$ be an atlas for M. (M m-dim.)

$$TM = \{(x, v) : x \in M, v \in T_x M\}.$$

For an atlas take $\{\tilde{U}_i, \tilde{\varphi}_i\}_{i \in I}$, where $\tilde{U}_i := \pi^{-1}(U_i)$ (here $\pi : TM \to M$ is the projection $\pi(x,v) = x$) and $\tilde{\varphi}_i : \tilde{U}_i \to \mathbb{R}^{2m}$ is defined by

$$\tilde{\varphi}_i(x,v) := (\varphi_i(x), T_x\varphi_i(v)).$$

Let $V_i := \varphi_i(U_i)$. The image of $\tilde{\varphi}_i$ is $V_i \times \mathbb{R}^m$, and $\tilde{\varphi}_i$ has inverse given by $\tilde{\varphi}_i^{-1} : V_i \times \mathbb{R}^m \to TM$ defined by

$$\tilde{\varphi}_i^{-1}(y, w) = (\varphi_i^{-1}(y), (T_x\varphi_i)^{-1}(w)).$$

Since φ_i is a homeomorphism, φ_i^{-1} is continuous, so clearly $\tilde{\varphi}_i$ and $\tilde{\varphi}_i^{-1}$ are continuous; so $\tilde{\varphi}_i$ is a homeomorphism. ✓

Clearly $\bigcup_{i\in I} \tilde{U}_i = TM$ since $\bigcup_{i \in I} U_i = M$. ✓

Let $\tilde{\varphi}_i$ and $\tilde{\varphi}_j$ be given. We NTS

$$\tilde{\varphi}_j \circ \tilde{\varphi}_i^{-1} : \underset{\varphi_i(U_i \cap U_j) \times \mathbb{R}^m}{\underset{\parallel}{\tilde{\varphi}_i(\tilde{U}_i \cap \tilde{U}_j)}} \longrightarrow \underset{\varphi_j(U_i \cap U_j) \times \mathbb{R}^m}{\underset{\parallel}{\tilde{\varphi}_j(\tilde{U}_i \cap \tilde{U}_j)}}$$

is smooth.

We have the explicit formula

$$\tilde{\varphi}_j \circ \tilde{\varphi}_i^{-1} : (y, w) \mapsto (\varphi_i^{-1}(y), T_y\varphi_i^{-1}(w)) \mapsto \left(\varphi_j \circ \varphi_i^{-1}(y), T_{\varphi_i^{-1}(y)}\varphi_j\left(T_y\varphi_i^{-1}(w)\right)\right).$$

The first coordinate is smooth because M is a mfld.

The second coordinate is smooth because it is essentially linear. ✓.

So TM is a smooth mfld w/ this atlas.

We want to show this atlas is orientable. That is, $T_{(y,w)}(\tilde{\varphi}_j \circ \tilde{\varphi}_i^{-1}) : \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ has positive determinant. Again, $\tilde{\varphi}_j \circ \tilde{\varphi}_i^{-1}$ is given by $(y,w) \mapsto (\varphi_j \circ \varphi_i^{-1}(y), T_y(\varphi_j \circ \varphi_i^{-1})(w))$.

$$T_{(y,w)}(\tilde{\varphi}_j \circ \tilde{\varphi}_i^{-1})(A, B) = \frac{d}{dt}\Big|_{t=0} \left(\varphi_j \circ \varphi_i^{-1}(\alpha(t)), T_{\alpha(t)}(\varphi_j \circ \varphi_i^{-1})(w+tB) \right)$$

where α is a path in M s.t. $\alpha(0) = y$ and $\alpha'(0) = A$

(because $T_{(y,w)}(\tilde{\varphi}_j \circ \tilde{\varphi}_i^{-1})(A, w) = \frac{d}{dt}\Big|_{t=0} (\tilde{\varphi}_j \circ \tilde{\varphi}_i^{-1})(\Lambda(t))$ —

where $\Lambda(t) := (\alpha(t), w + tB)$ is a path in $\varphi_i(U_i \cap U_j) \times \mathbb{R}^m$ w/ $\Lambda(0) = (y, w)$ and $\Lambda'(0) = (A, B)$ — $=$

$$= \frac{d}{dt}\bigg|_{t=0} \left(\varphi_j \circ \varphi_i^{-1}(\alpha(t)),\ T_{\alpha(t)}(\varphi_j \circ \varphi_i^{-1})(w + tB) \right).$$

So $T_{(y,w)}(\varphi_j \circ \varphi_i^{-1})(A,B) = \left(T_y(\varphi_j \circ \varphi_i^{-1})(A),\ \frac{d}{dt}\Big|_{t=0} T_{\alpha(t)}(\varphi_j \circ \varphi_i^{-1})(w + tB) \right).$

~~The (i,j) - component (not to be confused w/ the i,j of φ_j & φ_i) $(i,j = 1,\dots,m)$ of the Jacobian J is when $A = e_i$ and $B = 0$.~~

~~$J_{ij} =$~~

The Jacobian is $J = \left(\begin{array}{c|c} |||| & |||| \\ \hline |||| & |||| \end{array}\right)$.

The j^{th} column (not to be confused w/ the j of $\varphi_j \circ \varphi_i^{-1}$) $(j = 1,\dots,m)$ of the 1-1 quadrant equals

$$T_y(\varphi_j \circ \varphi_i^{-1})(e_j) = \left(j^{th} \text{ column of the Jacobian of } \varphi_j \circ \varphi_i^{-1}\right).$$

The j^{th} column of the 1-2 quadrant equals $T_y(\varphi_j \circ \varphi_i^{-1})(0) = 0$.

The j^{th} column of the 2-2 quadrant is evaluated w/ $\alpha(t) = y$ the constant path, as $A = 0$; so it just becomes $\frac{d}{dt}\Big|_{t=0} T_y(\varphi_j \circ \varphi_i^{-1})(w + te_j) = T_y(\varphi_j \circ \varphi_i^{-1})(e_j) =$

$= \left(\begin{matrix} j^{th} \text{ column of the Jacobian} \\ \text{of } \varphi_j \circ \varphi_i^{-1} \end{matrix} \right)$. Therefore,

$$\text{Det } J = \text{Det}\Big(\text{Jacobian of } \underbrace{\varphi_j \circ \varphi_i^{-1}}_{\text{invertible}}\Big)^2 > 0.$$

we conclude $\{\tilde{U}_i, \tilde{\varphi}_i\}_{i \in I}$ is an orientable atlas for TM. $\square$.

(b) Show the cotangent bundle T^*M is an orientable manifold.

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solution to (b). Again $\dim M = m$. $\{U_i, \varphi_i\}_{i \in I}$ is a chart.

$T^*M = \{(X, u) : X \in M \text{ and } u : T_X M \to \mathbb{R} \text{ linear}\}$.

We define an atlas $\{\tilde{U}_i, \tilde{\varphi}_i\}_{i \in I}$ on T^*M as follows: $\tilde{U}_i := \pi^{-1}(U_i)$ where $\pi : T^*M \to M$ is defined by $\pi(X, u) = X$, and $\tilde{\varphi}_i : \tilde{U}_i \to \mathbb{R}^{2m}$ is defined by $\tilde{\varphi}_i(X, u) = (\varphi_i(X), (u(\partial_1^i(X)), \ldots, u(\partial_m^i(X))))$

where $\partial_j^i(X) := T_{\varphi_i(X)}(\varphi_i^{-1})(e_j)$. we construct the inverse of $\tilde{\varphi}_i$. Since a linear form u is determined by its image on the basis $\{\partial_1^i(x), \dots, \partial_m^i(x)\}$, for any $w \in \mathbb{R}^m$ let u_w^i be the linear form $T_xM \to \mathbb{R}$ defined by $u_w^i(\partial_j^i(x)) = w_j$. Then the inverse $\tilde{\varphi}_i^{-1} : \tilde{\varphi}_i(\tilde{U}_i) \times \mathbb{R}^m \to T^*M$ of $\tilde{\varphi}_i$ is defined by $\tilde{\varphi}_i^{-1}(y, w) = (\varphi_i^{-1}(y), u_w^i)$. Clearly $\tilde{\varphi}_i$ and $\tilde{\varphi}_i^{-1}$ are continuous. ✓. (Note: The topology on T^*M has a basis given by $\{ {}^{-1}(U) : U \subset M \text{ open} \}$.)

Clearly $T^*M = \bigcup_{i \in I} \tilde{U}_i$ since $M = \bigcup_{i \in I} U_i$. ✓

Lastly we need $\tilde{\varphi}_j \circ \tilde{\varphi}_i^{-1} : \tilde{\varphi}_i(\tilde{U}_i \cap \tilde{U}_j) \times \mathbb{R}^m \to$ $\to \tilde{\varphi}_j(\tilde{U}_i \cap \tilde{U}_j) \times \mathbb{R}^m$ to be smooth. The formula for $\tilde{\varphi}_j \circ \tilde{\varphi}_i^{-1}$ is given by $(y, w) \mapsto (\varphi_i^{-1}(y), u_w^i) \mapsto$ $\mapsto \left(\varphi_j \circ \varphi_i^{-1}(y), \left(u_w^i(\partial_1^j(\varphi_i^{-1}(y))), \dots, u_w^i(\partial_m^j(\varphi_i^{-1}(y))) \right) \right)$ which is smooth since the first coordinate $\varphi_j \circ \varphi_i^{-1}(y)$ is smooth and the second coordinate is essentially linear. ✓.

We now want to show the Jacobian J of $\tilde{\varphi}_j \circ \tilde{\varphi}_i^{-1}$ has positive determinant. INCOMPLETE.