Geometry/Topology Qualifying Exam

Spring 2011

Solve all **SIX** *problems. Partial credit will be given to partial solutions.*

1. (10 pts) Let $S^3 = \{x \in \mathbb{R}^4 \mid \|x\| = 1\}$ be the 3-dimensional sphere, oriented as the boundary of the unit ball B^4 in $\mathbb{R}^4$ with the standard orientation. Compute $\int_{S^3} \omega$, where
$$\omega = x_1 dx_2 \wedge dx_3 \wedge dx_4 + x_2 dx_1 \wedge dx_3 \wedge dx_4 + x_3 dx_1 \wedge dx_2 \wedge dx_4.$$
(You may leave your answer in terms of volumes $vol(S^n)$ and $vol(B^n)$.)

2. (10 pts) Let $M = \{(x, y) \mid x, y \in \mathbb{R}^3, \|x\| = 1, \|y\| = 1, \langle x, y\rangle = 0\}$, where $\langle x, y\rangle$ is the standard inner product on $\mathbb{R}^3$. Show that M is a smooth compact embedded submanifold of $\mathbb{R}^6$ and explain how M can be identified with the unit tangent bundle of S^2.

3. (20 pts) Let $\mathbb{RP}^n$ be the real projective space given by $S^n/\sim$, where $S^n = \{\|x\| = 1\} \subset \mathbb{R}^{n+1}$ and $x \sim -x$ for all $x \in S^n$.
   (a) (5 pts) Use covering spaces to compute $\pi_1(\mathbb{RP}^n)$.
   (b) (5 pts) Give a cell (CW) decomposition of $\mathbb{RP}^n$ for $n \geq 1$.
   (c) (5 pts) Use the cell decomposition to compute the homology groups $H_k(\mathbb{RP}^n)$, $k \geq 0$.
   (d) (5 pts) For which values of $n \geq 1$ is $\mathbb{RP}^n$ orientable? Explain.

4. (10 pts) Given a continuous map $f : X \to Y$ between topological spaces, define
$$C_f = \Big((X \times [0, 1]) \coprod Y\Big) / \sim,$$
where $(x, 1) \sim f(x)$ for all $x \in X$ and $(x, 0) \sim (x', 0)$ for all $x, x' \in X$. Here $\coprod$ is the disjoint union. Then prove that there is a long exact sequence
$$\cdots \to H_{i+1}(X) \xrightarrow{f_*} H_{i+1}(Y) \to \widetilde{H}_{i+1}(C_f) \to H_i(X) \xrightarrow{f_*} H_i(Y) \to \cdots,$$
where f_* is the map on homology induced from f and $\widetilde{H}_i$ denotes the ith reduced homology group.

5. (10 pts) Prove that the fundamental group of a connected Lie group G is abelian. (A *Lie group* G is a smooth manifold which is also a group, and whose group operations multiplication and inverse are smooth maps.) [Hint: One possible way of proving this is to find an explicit homotopy between fg and gf, where f and g are loops in G.]

6. (10 pts) Let $M \subset \mathbb{R}^3$ be an embedded compact oriented surface (without boundary) of genus $g \geq 1$. Show that the Gaussian curvature κ of M must vanish somewhere on M.

Geometry Spring 2011

(1) Let $S^3 = \{ x \in \mathbb{R}^4 : \|x\| = 1 \}$ oriented as the boundary of $B^4 = \{ x \in \mathbb{R}^4 : \|x\| \leq 1 \}$ w/ the standard orientation.

Compute $\int_{S^3} \omega$ where

$$\omega = x_1 dx_2 \wedge dx_3 \wedge dx_4 + x_2 dx_1 \wedge dx_3 \wedge dx_4 + x_3 dx_1 \wedge dx_2 \wedge dx_4$$

in terms of $\mathrm{vol}(S^3)$ and $\mathrm{vol}(B^4)$.

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solution

$$\int_{S^3} \omega = \int_{B^4} d\omega .$$

$v := dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$ standard volume form on $\mathbb{R}^4$.

$\mathrm{vol}(B^4) := \int_{B^4} v$.

$$d\omega = v - v + v = v.$$

$$\int_{S^3} \omega = \mathrm{vol}(B^4).$$

$\square$

(2) $M = \{ (x,y) : x,y \in \mathbb{R}^3,\ \|x\|=1,\ \|y\|=1,\ \langle x,y\rangle = 0 \}$

where $\langle x,y \rangle$ is the standard inner product in $\mathbb{R}^3$.

Show M is a smooth compact embedded submanifold of $\mathbb{R}^6$, and show M can be identified w/ the unit tangent bundle of S^2.

<u>solution</u> $f : \mathbb{R}^6 \longrightarrow \mathbb{R}^3$

$$f(x_1,x_2,x_3,y_1,y_2,y_3) = (x_1^2+x_2^2+x_3^2,\ y_1^2+y_2^2+y_3^2,\ x_1y_1+x_2y_2+x_3y_3).$$

$$M = f^{-1}(\{(1,1,0)\}).$$ Let $(x,y) \in M$.

$$T_{(x,y)}f = \begin{pmatrix} 2x_1 & 2x_2 & 2x_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2y_1 & 2y_2 & 2y_3 \\ y_1 & y_2 & y_3 & x_1 & x_2 & x_3 \end{pmatrix}$$

~~Let~~ $\cancel{\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3}$

~~Let~~ $\cancel{\begin{pmatrix} v_1 \\ \vdots \\ v_6 \end{pmatrix} \in \mathbb{R}^6}$.

~~Want to solve~~ $\cancel{\begin{pmatrix} 2x_1v_1+2x_2v_2+2x_3v_3 \\ 2y_1v_4+2y_2v_5+2y_3v_6 \\ v_1y_1+v_2y_2+v_3y_3+v_4x_1+v_5x_2+v_6x_3 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}}$

Take $\cancel{(v_1,v_2,v_3) = \frac{a}{2}(x_1,x_2,x_3)}$ and

Since $|x|^2=1$ and $|y|^2=1$, the first and 2nd rows are l.i. Since $\langle x,y\rangle=0$, the third row is l.i. from the first two rows. Therefore, the matrix has rank 3 and M is an embedded submanifold of $\mathbb{R}^6$ of dimension 3. M is closed and contained in the 5-sphere of radius

$\sqrt{2}$ centered at the origin in $\mathbb{R}^6$; so M is bounded and closed, hence compact.

The unit tangent bundle of S^2

is $T_1S^2 = \{(X,y) : X \in S^2, y \in T_XS^2 \text{ and } \|y\|^2 = 1\}$ where we have identified $T_XS^2 \cong \ker T_X g \subset \mathbb{R}^3$ as the kernel of the derivative $T_X g : \mathbb{R}^3 \to \mathbb{R}$ of the function $g : \mathbb{R}^3 \to \mathbb{R}$, $g(X) = \|X\|^2$ which defines $S^2 = g^{-1}(\{1\})$ (Here, T_XS^2 is the equivalence classes of paths starting at X where two paths are equivalent if they have the same initial velocity; this initial velocity is an elt of $\ker T_X g$.)

Observe $T_X g = (2X_1 \; 2X_2 \; 2X_3)$ so (y_1, y_2, y_3) is in the kernel of $T_X g$ iff $\langle X, y \rangle = 0$. Hence $T_1S^2 \cong \{(X,y) : X \in S^2, y \in \mathbb{R}^3, \|y\| = 1, \langle X, y \rangle = 0\} = M$. $\square$

(3) Let $\mathbb{R}P^n$ be real projective space, realized as a topological space as $S^n/\sim$ where $x \sim -x$ and $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$.

(a) Use covering spaces to compute $\pi_1(\mathbb{R}P^n)$.

(b) Give a cell (CW) decomposition of $\mathbb{R}P^n$ for $n \geq 1$.

(c) Use the cell decomposition to compute the homology groups $H_k(\mathbb{R}P^n)$, $k \geq 0$.

(d) For which values of $n \geq 1$ is $\mathbb{R}P^n$ orientable? Explain.

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solution to (a)

$\mathbb{R}P^1 \cong S^1$ so $\pi_1(\mathbb{R}P^1) = \mathbb{Z}$.

For $n \geq 2$, S^n is simply connected and is a covering space of $\mathbb{R}P^n$ w/ covering map $p : S^n \to \mathbb{R}P^n$ defined by $x \mapsto \{x, -x\}$.

so S^n is the universal cover of $\mathbb{R}P^n$.

so $\pi_1(\mathbb{R}P^n) \cong \mathrm{Aut}(S^n, p) := \{\varphi : S^n \xrightarrow[\text{homeo}]{} S^n \text{ s.t. } p \circ \varphi = p\}$.

Claim: $\varphi \in \mathrm{Aut}(S^n, p)$ iff $\varphi(x) = x, \forall x \in S^n$ or $\varphi(x) = -x, \forall x \in S^n$.

Proof (claim): Define $f : S^n \to \mathbb{R}$ by $f(x) = +1$ if $\varphi(x) = x$ and -1 if $\varphi(x) = -x$. Clearly f is continuous. Then $S^n = f^{-1}(1) \cup f^{-1}(-1)$ is a decomp of S^n into two closed-open subspace; since S^n is connected, one must be empty. ✓.

We conclude $\mathrm{Aut}(S^n, p) \cong \mathbb{Z}/2\mathbb{Z} \cong \pi_1(\mathbb{R}P^n), n \geq 2$.

$\square$.

<u>solution to (b)</u>

We can give it for $\mathbb{R}P^0$ too.

$\mathbb{R}P^0 = \{\text{lines in } \mathbb{R}^1\} = \{*\}$ a point.

<u>$n \geq 1$</u>: $\mathbb{R}P^n = S^n/\sim$ where $x \sim -x$ $\quad (S^n \subset \mathbb{R}^{n+1})$

$= D^n/\sim$ where $x \sim -x$, $x \in \partial D^n = S^{n-1}$ $\quad \left(D^n \subset S^n \text{ closed upper hemisphere}\right)$

$= D^n/\mathbb{R}P^{n-1}$.

Hence we see that $\mathbb{R}P^n$ has a cell decomposition $e^0 \cup e^1 \cup \cdots \cup e^n$, where for $1 \leq i \leq n$ the i^{th} attaching map φ_i attaches the boundary S^{i-1} of $\overline{e^i} = D^i$ to $\mathbb{R}P^{i-1}$ by identifying antipodal points.

More rigorously, define $\mathbb{R}P^n$ inductively, where for $n \geq 1$, $\mathbb{R}P^n := \mathbb{R}P^{n-1} \cup_{\varphi_n} D^n$ where as a set $\mathbb{R}P^{n-1} = S^{n-1}/\sim$, $S^{n-1} = \partial D^n$, and where $\varphi_n : \partial D^n \to \mathbb{R}P^{n-1}$, $x \overset{\varphi_n}{\mapsto} [x]$.

$\square$.

solution to (c)

$$\mathbb{R}P^n = X^{(0)} \cup X^{(1)} \cup \dots \cup X^{(n)}$$

where $X^{(k+1)} = X^{(k)} \amalg_{\varphi_k} B^{k+1}$

where B^{k+1} = closed unit ball in $\mathbb{R}^{k+1}$

and $\varphi_k : S^k = \partial B^{k+1} \longrightarrow X^{(k)} \cong \mathbb{R}P^k$

is defined by $\varphi_k(x) = \phi_k(\{x, -x\})$.

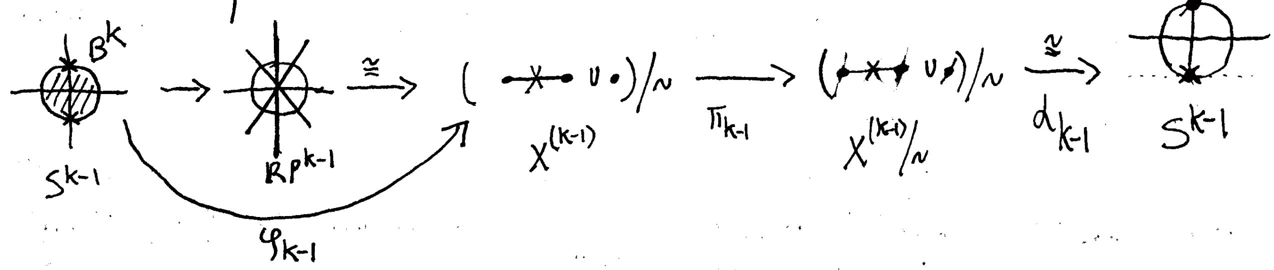
Choose some iso $\phi_k : \mathbb{R}P^k \to X^{(k)}$ by induction.

Let $\pi_k : X^{(k)} \to X^{(k)}/\sim \cong S^k$ be defined by collapsing $X^{(k-1)}$ to a point.

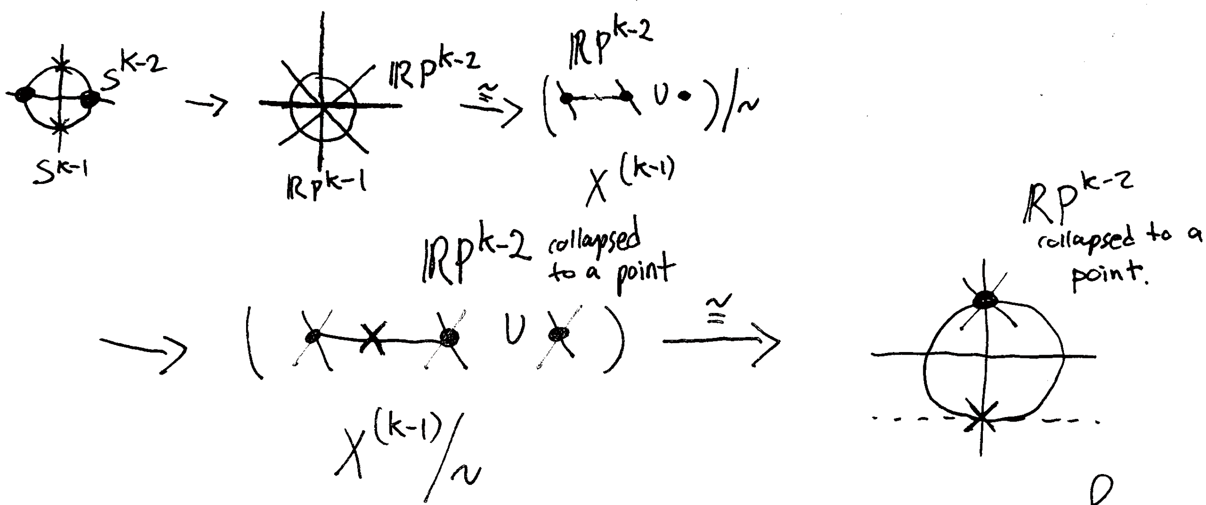
We wish to compute the degree of

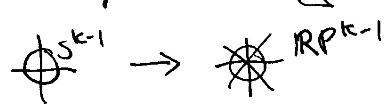
$$S^{k-1} \xrightarrow{\varphi_{k-1}} X^{(k-1)} \xrightarrow{\pi_{k-1}} X^{(k-1)}/\sim \xrightarrow[\alpha_{k-1}]{\cong} S^{k-1}$$

where α_{k-1} is the standard iso sending a ball modulo its boundary to a sphere of the same dimension.



Let $S^{k-2} \subset S^{k-1} \subset \mathbb{R}^k$ be the set of points in S^{k-1} w/ zero last coordinate. The above pictures become



we see that the composite $S^{k-1} \xrightarrow{f_{k-1}} S^{k-1}$ stretches each (open) upper/lower hemisphere homeomorphically onto $S^{k-1} - \{\text{north pole}\}$. Let f^N_{k-1}, f^S_{k-1} be the corresponding homeos. Because the first map defining the composite $S^{k-1} \to S^{k-1}$ is 

, clearly $f^N_{k-1} = f^S_{k-1} \circ r$ or $f^S_{k-1} = f^N_{k-1} \circ r$, where r is the antipodal map $S^{k-1} \to S^{k-1}$, having degree $(-1)^{k-1}$ (restricting r to upper/lower (open) hemisphere, respectively.)

~~Let $[\sigma^N - \sigma^S]$ be the generator of $H_{k-1}(S^{k-1})$ where $\sigma^N/\sigma^S : \Delta^{k-1} \to S^{k-1}$ map Δ^{k-1} homeomorphically (rel boundary) to the closed upper/lower hemisphere, respectively.~~

~~Then $H_{k-1}(S^{k-1} \xrightarrow{f} S^{k-1})([\sigma^N - \sigma^S])$~~

$$\cancel{= [f \circ \sigma^N - f \circ \sigma^S]}$$

$$\cancel{= [f \circ \sigma^N - f \circ r \circ \sigma^N]}$$

$$\cancel{= [f^N_{k-1} \circ \sigma^N}$$

Let U_N and U_S be the open upper and lower hemispheres, respectively, and let V be the sphere minus the north pole.
We want to compute the local degrees of f_{k-1} w.r.t. U_N and U_S. More specifically, observe that if x_N = north pole and x_S = south pole, $x_S = y$, then $f^{-1}(y) = \{x_N, x_S\}$ (follow the green x's in the diagrams). Then we wish to compute the local degree of $H_{k-1}(S^{k-1}) \cong H_{k-1}(U_N, U_N - x_N) \xrightarrow{H_{k-1}(f_{k-1})} H_n(V, V-y) \cong H_{k-1}(S^{k-1})$ and

for the same w/ $U_N \leftrightarrow U_S$ and $X_N \leftrightarrow X_S$.

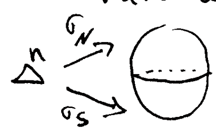
(For simplicity, for the time being let $n \equiv k-1$.) (And write $f \equiv f_{k-1}$.)

The iso $H_n(S^n) \cong H_n(U_N, U_N - X_N)$

is determined by the comm diagram (black only):

$$\begin{tikzcd}
 & H_n(U_N, U_N - X_N) \arrow[r, "H_n(f)"] \arrow[dl, "\sim"'] \arrow[d] & H_n(V, V-y) \arrow[d, "\wr"] \\
H_n(S^n, S^n - X_N) & H_n(S^n, S^n - f^{-1}(y)) \arrow[l] \arrow[r, "H_n(f)"] & H_n(S^n, S^n - y) \\
 & H_n(S^n) \arrow[u] \arrow[ul, "\sim"] \arrow[r, "H_n(f)"] & H_n(S^n) \arrow[u, "\wr"']
\end{tikzcd}$$

To find a generator for $H_n(U_N, U_N - X_N)$ we follow the generator $[\sigma_N - \sigma_S]$ for $H_n(S^n)$, where σ_N and σ_S are singular simplices mapping Δ^n homeomorphically to the closure of U_N and U_S in S^n, respectively, and in a natural way s.t. boundaries are preserved and σ_N 'matches' 'correctly' w/ σ_S:

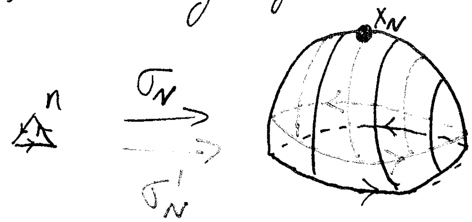


The map $H_n(S^n) \to H_n(S^n, S^n - X_N)$ generally sends $\underset{\substack{\partial c = 0 \\ c \in C_n(S^n)}}{[c]} \longmapsto [c]$. And the map $H_n(U_N, U_N - X_N) \longrightarrow H_n(S^n, S^n - X_N)$ generally sends $\underset{\substack{c \in C_n(U_N) \\ \partial c \in C_{n-1}(U_N - X_N)}}{[c]} \mapsto [c]$

So $[\sigma_N - \sigma_S] \in H_n(S^n)$ is just mapped to $[\sigma_N - \sigma_S] \in H_n(S^n, S^n - x_N)$.

The harder part is finding the preimage of this elt in $H_n(U_N, U_N - x_N)$.

<u>claim 1</u>: Let $\sigma_N' \in C_n(U_N)$ be the singular simplex obtained by 'shrinking' σ_N just slightly to lie in U_N:



Clearly $\partial \sigma_N' \in C_{n-1}(S^n - x_N)$, so $[\sigma_N']$ is a w.d. elt of $H_n(U_N, U_N - x_N)$.

The claim is $[\sigma_N']$ is the preimage of $[\sigma_N - \sigma_S]$.

~~Proof (claim): we need to show~~

$\cancel{[\sigma_N'] = [\sigma_N - \sigma_S] \in H_n(S^n, S^n - x_N)}$, ~~i.e.~~

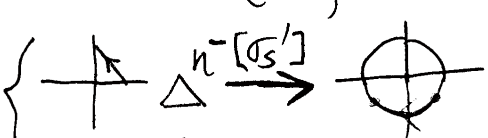
Proof (claim 2):

Omitted. $\square$.

(The claim is at least intuitively plausible — what else would generate $H_n(U_N, U_N - X_N)$? — and I did validate the claim in the case $n = 1$ (see below), but it involved calculating w/ simplices which can be complicated in higher dimensions.)

Cor/ $[\sigma_N']$ generates $H_n(U_N, U_N - X_N)$.

Proof (cor): $\square$ see claim 3!!! below



And see claim 4 for the def of σ_s.

Similarly, returning to the comm diagram above, $-[\sigma_s']$ generates $H_n(V, V - y)$ and corresponds to the generator $[\sigma_N - \sigma_S]$ of $H_n(S^n)$. So we wish to calculate $H_n(f)([\sigma_N'])$ in terms of $-[\sigma_s']$.

Regarding claim 1 when $n=1$:

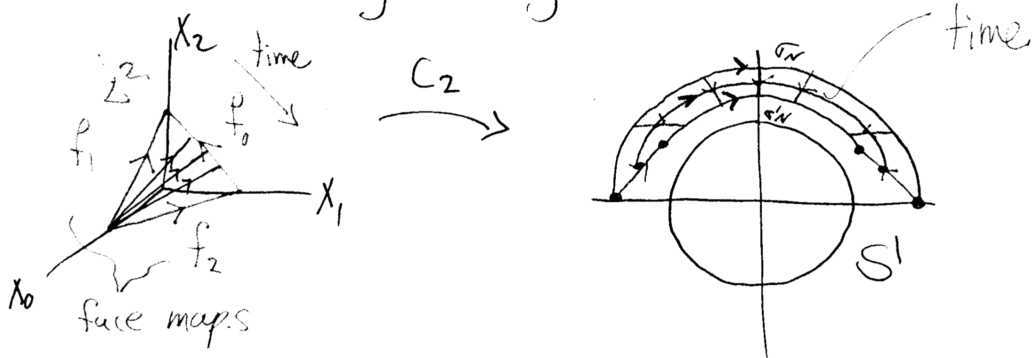
$$[\sigma_N'] = [\sigma_N - \sigma_S] \in H_n(S^n, S^n - x_N)$$

iff $\sigma_N' - \sigma_N + \sigma_S \in B_n(S^n, S^n - x_N)$

iff There exists $c_{n+1} \in C_{n+1}(S^n)$ s.t.

$$\partial c_{n+1} - \sigma_N' + \sigma_N - \sigma_S \in C_n(S^n - x_N).$$

When $n=1$, we construct $c_2 \in C_2(S^1)$ as the singular 2-simplex indicated by the following diagram:



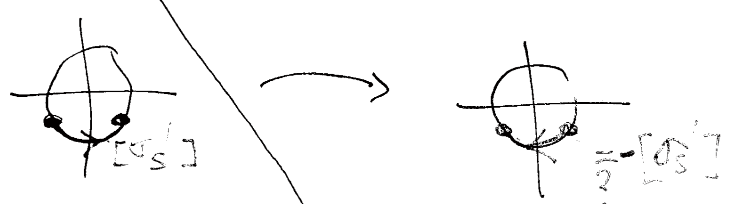
One calculates $\partial c_2 = \underbrace{c_2 \circ f_0}_{\in C_1(S^1 - x_N)} - \sigma_N + \sigma_N'$.

Claim 3 : $H_n(f)([\sigma_s']) = (-1)^n [\sigma_s'] \in H_1(V, V-y)$

Proof (claim 3):

The proof is relatively clear when $n=1$:

Again, following the usual pictures we see



Call the image $[\sigma]$. We wish to know

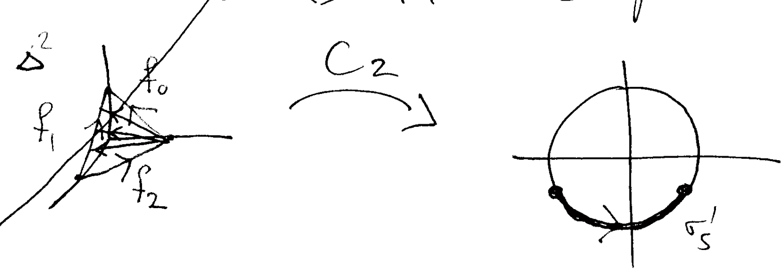
if $[\sigma] = -[\sigma_s']$ (intuitively, yes)

iff $\sigma + \sigma_s' \in B_1(V, V-y)$

iff There exists $c_2 \in C_2(V)$ s.t.

$$\partial c_2 - \sigma - \sigma_s' \in C_1(V-y).$$

To construct c_2, we use the same (almost) schematic as in the proof of claim 1:



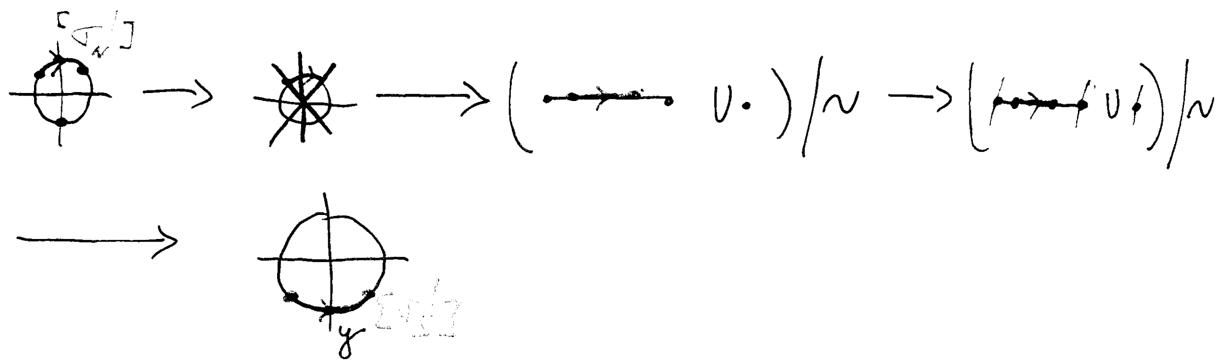
($\partial c_2 = \sigma + \sigma_s'$ 'time' is irrelevant unlike before.)

<u>claim 2</u> : $H_n(f)([\sigma_N']) = [\sigma_S'] \in H_n(V, V-y)$

(recall we really mean $n \equiv k-1$, $f \equiv f_{k-1}$)

<u>Proof (claim 2)</u> :

This is clear when $n = 1$:



we omit <u>the proof</u> for $n > 1$. $\square$

Hence the local degree $\deg f|_{X_N} = 1$.

The calculation of $\deg f|_{X_S}$ proceeds similarly: we wish to know where $H_n(f)$ takes the generator $-[\sigma_S']$ of $H_n(U_S, U_S - X_S)$ (corresponding to $[\sigma_N - \sigma_S] \in H_n(S^n)$) in terms of the generator $[\sigma_S']$ of $H_n(V, V-y)$ (also corresponding to $[\sigma_N - \sigma_S] \in H_n(S^n)$, same as before):

Claim 3: $-[\sigma_s']$ corresponds to $[\sigma_N - \sigma_S]$ in the isomorphism $H_n(U_s, U_s - X_s) \cong H_n(S^n)$
(or $H_n(V, V-y) \cong H_n(S^n)$)

Proof (claim 3):

The proof when $n=1$:

we show if σ is the 1-simplex  then

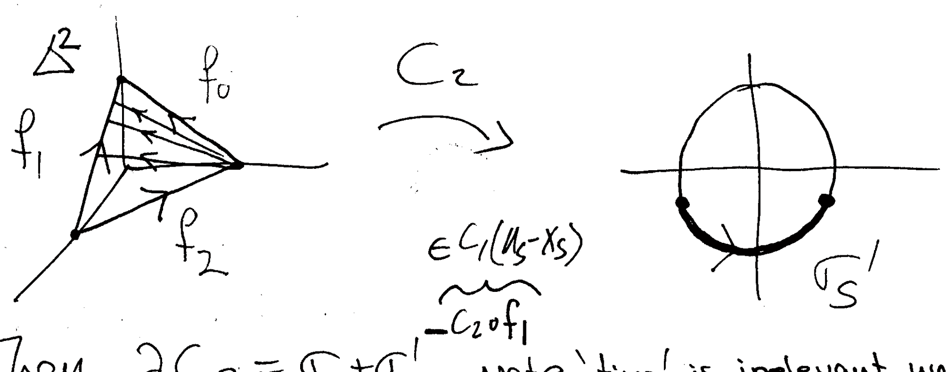
$$-[\sigma_s'] = [\sigma] \in H_1(U_s, U_s - X_s)$$

iff $\sigma + \sigma_s' \in B_1(U_s, U_s - X_s)$

iff There exists $c_2 \in C_2(U_s)$ s.t.

$$\partial c_2 - \sigma - \sigma_s' \in C_1(U_s - X_s).$$

We proceed using the same (almost) schematic as in the proof of Claim 1:

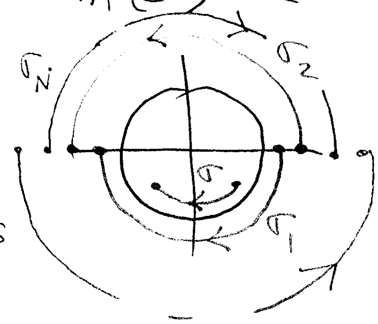


Then $\partial C_2 = \sigma + \sigma_s'$. Note 'time' is irrelevant, unlike before.

From claim 2, it follows that

$$[\sigma_1 - \sigma_2] \in H_1(S^1) \longleftrightarrow [\sigma] \in H_1(U_S, U_S - X_S)$$

where $[\sigma] = -[\sigma_S']$:



(turn the paper upside down).

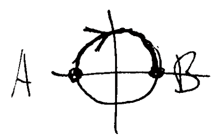
we will be done if $[\sigma_1 - \sigma_2] = [\sigma_N - \sigma_S] \in H_1(S^1)$.

That is, $\sigma_1 - \sigma_2 - \sigma_N + \sigma_S$ is the boundary ∂c_2 of some $c_2 \in C_2(S^1)$.

Indeed, similar to before, take c_2 as the difference of two simplices:

boundary
$\sigma_S - c_B + \sigma_1$
(c_B is the constant map at B)

minus

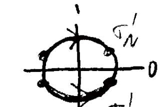


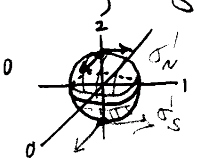
boundary
$\sigma_N - c_B + \sigma_2$.

we omit the proof of claim 3 for $n > 1$.
(The proof for $H_n(V, V-g)$ is clearly identical.) $\square$.

Claim 4: (Geometric description of $-[\sigma_S^1]$)

Define $r : S^n \to S^n$ to be the map sending the last coordinate to its negative. Then, all along in this solution, $\sigma_S^1 := r \circ \sigma_N^1$ where σ_N^1 is understood.

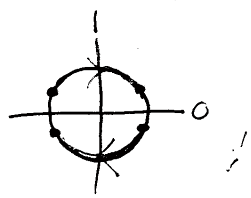




Define $r_n : S^n \to S^n$ by

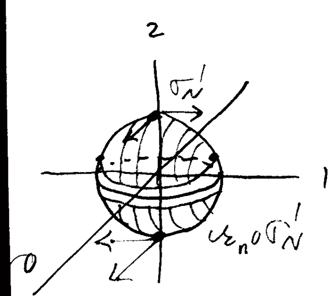
$$r_n(x_0, x_1, \dots, x_n) = ((-1)^n x_0, -x_1, -x_2, \dots, -x_n).$$

(So $r_n = \overset{\overset{\text{antipodal map } S^n \to S^n}{\equiv}}{r}$ when n is odd.)



Then $-[\sigma_S^1] = [r_n \circ \sigma_N^1] \in$

$H_n(U_S, U_S - x_S)$ or $\in H_n(V, V-y)$.



Moreover, the same is true if r_n is replaced by $r_n(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{n-1}) : S^n \to S^n$ defined by $(x_0, x_1, \dots, x_{n-1}, x_n) \mapsto (\varepsilon_0 x_0, \varepsilon_1 x_1, \dots, \varepsilon_{n-1} x_{n-1}, -x_n)$ where $\varepsilon_i = \pm 1$ and $\#\{i : \varepsilon_i = -1\}$ is odd.

Proof The proof for $n=1$ was done in the proof of Claim 3. We omit the proof for $n>1$. $\square$

The geometric content of claim 2 is that under f, σ_N' maps to σ_S'. The consequence, by claim 3, is that the local degree is -1.

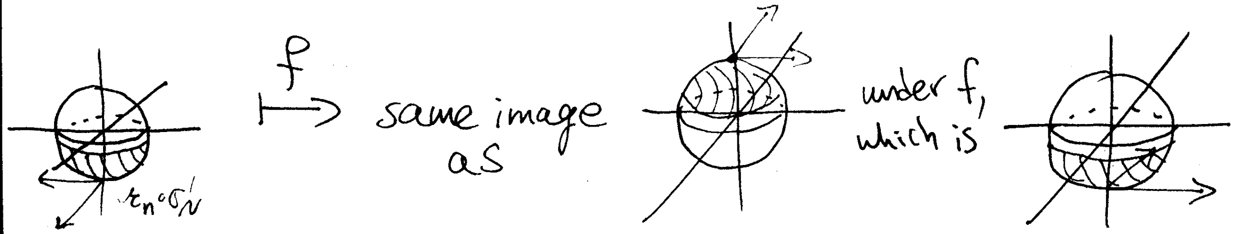
Observe that if n is odd, then by the definition of f, $r_n \circ \sigma_N' = r \circ \sigma_N'$ is sent to the same image as σ_N', hence

$$H_n(f)(-[\sigma_S']) = H_n(f)([r_n \circ \sigma_N'])$$

$$= H_n(f)([\sigma_N']) = [\sigma_S'] \in H_n(V, V-y),$$

by claims 2 & 4. Hence the local degree is again -1. Conversely,

<u>claim 5</u> : Let n be even. f maps $r_n \circ \sigma_N'$ to $r_n(-1, +1, \ldots, +1) \circ \sigma_N'$.



<u>Proof</u>: Obvious. $\square$.

Consequently, when n is even, by claim 4, $H_n(f)(-[\sigma_s'])= -[\sigma_s'] \in H_n(V, V-y)$.

Hence the local degree is $+1$.

<u>Summarizing</u>: The degree of f is (sum local degrees) $= -1 + (-1)$ when n is odd and $-1 + (+1)$ when n is even. Recall now $n \equiv k-1$ and $f \equiv f_{k-1}$.

By the main theorem of cellular homology we obtain the cellular complex for $\mathbb{RP}^n$

$$\cdots \longrightarrow \underbrace{H_k(X^{(k)}, X^{(k-1)})}_{\cong\, \mathbb{Z}} \xrightarrow{d_k} \underbrace{H_{k-1}(X^{(k-1)}, X^{(k-2)})}_{\cong\, \mathbb{Z}} \longrightarrow \cdots$$

where d_k is multiplication by the degree of f_{k-1}. So:

$$\begin{array}{lccccccccc}
 & n & n-1 & n-2 & & 3 & 2 & 1 & 0 & \\
n \text{ odd}: \xrightarrow{0} & \mathbb{Z} \xrightarrow{0} & \mathbb{Z} \xrightarrow{-2} & \mathbb{Z} \xrightarrow{0} & \cdots \xrightarrow{-2} & \mathbb{Z} \xrightarrow{0} & \mathbb{Z} \xrightarrow{-2} & \mathbb{Z} \xrightarrow{0} & \mathbb{Z} \longrightarrow & 0 \\
n \text{ even}: \xrightarrow{0} & \mathbb{Z} \xrightarrow{-2} & \mathbb{Z} \xrightarrow{0} & \mathbb{Z} \xrightarrow{-2} & \cdots \xrightarrow{-2} & \mathbb{Z} \xrightarrow{0} & \mathbb{Z} \xrightarrow{-2} & \mathbb{Z} \xrightarrow{0} & \mathbb{Z} \longrightarrow & 0
\end{array}$$

so $H_k(\mathbb{RP}^n) = \begin{cases} n \text{ odd}: \mathbb{Z}\ (k=0,n),\ \mathbb{Z}_2\ (k=1,3,\dots,n-2),\ 0\ (k=2,4,n-1) \\ n \text{ even}: \mathbb{Z}\ (k=0),\ \mathbb{Z}_2\ (k=1,3,\dots,n-1),\ 0\ (k=2,4,\dots,n). \end{cases}$ $\square$

solution to (d)

We want to answer the question

When is $\mathbb{R}P^n$ orientable?

topological mfld of dim n ✓

Since $\mathbb{R}P^n$ is a compact and connected,

$$\begin{cases} \text{if orientable, then } H_n(\mathbb{R}P^n) = \mathbb{Z} \\ \text{if \underline{not} orientable, then } H_n(\mathbb{R}P^n) = 0 . \end{cases}$$

we conclude $\mathbb{R}P^n$ is orientable when n is odd, and is not orientable when n is even.

sanity check: $\mathbb{R}P^1 \cong S^1$ orientable. ✓

$\mathbb{R}P^2$ is not orientable.

$\square$.

(4) Let $f : X \to Y$ be a continuous map between top spaces. Define

$$C_f = \big((X \times [0,1]) \amalg Y \big) / \sim$$

where $(x,1) \sim f(x)$ for all $x \in X$ and $(x,0) \sim (x',0)$ for all $x, x' \in X$.

Prove there is a LES

$$\cdots \to H_{k+1}(X) \xrightarrow{H_{k+1}(f)} H_{k+1}(Y) \to \tilde{H}_{k+1}(C_f) \to H_k(X) \xrightarrow{H_k(f)} H_k(Y) \to$$

where $\tilde{H}_k(C_f)$ is the k^{th} reduced homology group.

<u>solution</u>

Let $X' := f(X) \overset{i}{\subset} Y$. Then we have a LES in relative homology

$$\to H_{k+2}(Y,X') \xrightarrow{\delta_{k+2}} H_{k+1}(X') \xrightarrow{H_{k+1}(i)} H_{k+1}(Y) \to H_{k+1}(Y,X') \xrightarrow{\delta_{k+1}} H_k(X') \xrightarrow{H_k(i)} H_k(Y) \to$$

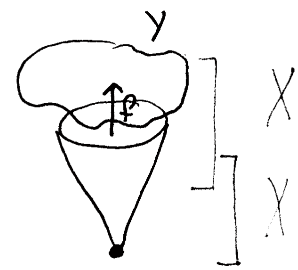
$\delta_{k+1}([c]) = [\partial c]$.

$\partial c \in C_k(X')$

solution

Let $X_2 = ((X \times [\frac{1}{4}, 1]) \amalg Y) / \sim$

and $X_1 = (X \times [0, 3/4]) / \sim$



Clearly $X_1 \cap X_2$ is a nbhd def retract in X_1 and X_2, and $X_1, X_2 \subset C_f$ are closed.

So we get a Mayer-Vietoris LES

$$\to H_k(X_1 \cap X_2) \to H_k(X_1) \oplus H_k(X_2) \to H_k(C_f) \to H_{k-1}(X_1 \cap X_2) \to$$

Clearly $H_k(X_1 \cap X_2) \cong H_k(X)$ and

$$H_k(X_1) \oplus H_k(X_2) \cong \begin{cases} H_k(Y), & k > 0 \\ \mathbb{Z} \oplus H_0(Y), & k = 0 \end{cases}$$

Since everything is a def retract we need to make sense of these iso's as well as the claim $H_k(X) \to H_k(Y)$ is the map $H_k(f)$.

If $\overset{\text{a subspace}}{\vee}$ A is a deformation retract of a space X, then there is an isomorphism $H_k(A) \xrightarrow{\sim} H_k(X)$ which is induced by the embedding $A \xrightarrow{i} X$ and so has the formula $\underset{\substack{\sigma \in C_k(A) \text{ simplex} \\ \partial\sigma = 0}}{[\sigma]} \mapsto [i \circ \sigma] \in H_k(X)$.

Conversely, if $X \xrightarrow{p} A$ is the retraction at the end of the def retract, then there is an iso $H_k(X) \xrightarrow{\sim} H_k(A)$ induced by p and so has the formula $\underset{\substack{\sigma \in C_k(X) \text{ simplex} \\ \partial\sigma = 0}}{[\sigma]} \mapsto [p \circ \sigma]$.

Identify X w/ $X \times \{\frac{1}{2}\}$ and let p_k be the retract $X_1 \cap X_2 \xrightarrow{p_k} X$ defined by $(x,t) \overset{p_k}{\mapsto} (x, \frac{1}{2})$. So we have an iso $H_k(X_1 \cap X_2) \xrightarrow{H_k(p_k)} H_k(X)$. Let i_k be the embedding $X \to X_1 \cap X_2$ inducing an iso in the reverse direction.

Let q_k be the retraction $X_2 \to Y$ defined by $(x,t) \overset{q_k}{\mapsto} f(x)$ and $y \overset{q_k}{\mapsto} y$, inducing an iso $H_k(X_2) \xrightarrow{H_k(q_k)} H_k(Y)$. Let j_k be the reverse inclusion $Y \xrightarrow{j_k} X_2$.

Let r_K be the retraction $X_1 \longrightarrow \{p\}$, ...

where p is the point $X \times \{0\} \in C_f$, defined by $(x,t) \mapsto (x,0) = p$, which induces an iso $H_k(X_1) \xrightarrow{H_k(r_k)} H_k(p)$. Let l_k be the reverse inclusion $p \xrightarrow{l_k} X_1$.

Let a_1 and a_2 be the inclusions $X_1 \cap X_2 \xrightarrow{a_1} X_1$ and $X_1 \cap X_2 \xrightarrow{a_2} X_2$. Let b_1 and b_2 be the inclusions $X_1 \xrightarrow{b_1} C_f$ and $X_2 \xrightarrow{b_2} C_f$. Recall the Mayer-Vietoris maps are given by

$$\to H_k(X_1 \cap X_2) \xrightarrow{(-H_k(a_1)) \oplus (+H_k(a_2))} H_k(X_1) \oplus H_k(X_2) \xrightarrow{H_k(b_1) + H_k(b_2)} H_k(C_f) \rightsquigarrow H_{k-1}(X_1 \cap X_2) \to$$

Using these various maps, we may construct a sequence

$$\begin{tikzcd}
\to H_{k+1}(C_f) \arrow[r] \arrow[dr, dashed] & H_k(X_1 \cap X_2) \arrow[r] \arrow[d, shift right, "\wr"'] & H_k(X_1) \oplus H_k(X_2) \arrow[r] \arrow[d, shift right, "\wr"'] & H_k(C_f) \to \arrow[dr, dashed] & \\
& H_k(X) \arrow[r, dashed] \arrow[u, shift right, "\wr"'] & H_k(p) \oplus H_k(Y) \arrow[u, shift right, "\wr"'] \arrow[ur, dashed] & & {}
\end{tikzcd}$$

from which we get

$$\to H_{k+1}(C_f) \to H_k(X) \to H_k(p) \oplus H_k(Y) \to H_k(C_f) \to$$

which is clearly exact since we have done nothing. Now, when $K > 0$, $H_k(p) \equiv 0$, so the map $H_k(X) \to H_k(p) \oplus H_k(Y) \equiv H_k(Y)$ is given by the composition

$$H_k(X) \xrightarrow{H_k(i_k)} H_k(X_1 \cap X_2) \xrightarrow{(-H_k(a_1)) \oplus H_k(a_2))} H_k(X_1) \oplus H_k(X_2) \xrightarrow{H_k(r_k) \oplus H_k(l_k)} H_k(p) \oplus H_k(Y) \equiv H_k(Y)$$

given by the formula

$$[\sigma] \mapsto [i_k \circ \sigma] \mapsto \left(-[a_1 \circ i_k \circ \sigma], [a_2 \circ i_k \circ \sigma]\right) \mapsto [q_k \circ a_2 \circ i_k \circ \sigma].$$

$\sigma \in C_k(X)$
$\partial\sigma = 0$

And the simplex $q_k \circ a_2 \circ i_k \circ \sigma$ has the formula

$$(x, \tfrac{1}{2}) \mapsto (x, \tfrac{1}{2}) \mapsto (x, \tfrac{1}{2}) \mapsto f(x)$$

So we see the map $H_k(X) \longrightarrow H_k(Y)$ is indeed the map $H_k(f)$ for $k > 0$.

When $k = 0$, we have the part of the LES

$$\to H_0(X) \to \overbrace{H_0(p)}^{\cong \mathbb{Z}} \oplus H_0(Y) \longrightarrow H_0(C_f) \longrightarrow 0 \longrightarrow$$

We want the end of our LES to look like

$$\to H_0(X) \to H_0(Y) \longrightarrow \underbrace{\tilde{H}_0(C_f)}_{\cong \mathbb{Z}^{(\operatorname{rank} H_0(C_f) - 1)}} \longrightarrow 0 \longrightarrow$$

Let r_X be the number of components of X so that $\{[x_i]_{i=1,\dots,r_X}\}$ is a basis of $H_0(X)$ for any $x_i \in X_i$ the i^{th} component of X. The map $H_0(X) \to H_0(p) \oplus H_0(Y)$ is given by $\sum_{i=1}^{r_X} n_i[x_i] \longmapsto \left(-\sum_{i=1}^{r_X} n_i[p], \sum_{i=1}^{r_X} n_i[f(x_i)]\right)$.

(Let $y_0 \in Y$ be in a component containing a piece of the image $f(X)$. Then since $(X \times [0,1])/\sim$ is path connected in C_f, we may

~~rewrite the above map~~ $\cancel{H_0(X) \to H_0(p) \oplus H_0(Y)}$ ~~as~~

$$\cancel{\sum_{i=1}^{r_X} n_i[X_i] \mapsto \left(-\sum_{i=1}^{r_X} n_i[p], \sum_{i=1}^{r_X} n_i[y_0]\right)}$$

---

write $[f(X)] = [y_0] \in H_0(C_f)$ for all $X \in \mathcal{X}$. This parenthetical comment will have relevance shortly.)

Let r_Y be the number of components of Y. For any $y_i \in Y_i$, the i^{th} component of Y, $i = 1, \ldots, r_Y$, $\{[y_i]\}_{i=1,\ldots,r_Y}$ is a basis for $H_0(Y)$. Let $X_i, i=1,\ldots,r_X$, be as before. For each $i = 1, \ldots, r_Y$ let $A_i = \{X_j : j = 1, \ldots, r_X \text{ and } f(X_j) \in Y_i\}$. Then the map $H_0(X) \to H_0(p) \oplus H_0(Y)$ from before has the formula

$$\sum_{j=1}^{r_X} n_j [X_j] \mapsto \left(-\sum_{j=1}^{r_X} n_j [p], \sum_{i=1}^{r_Y} \Big(\sum_{j \in A_i} n_j\Big)[y_i]\right).$$

This is the same as the function $\mathbb{Z}^{r_X} \xrightarrow{F} \mathbb{Z} \oplus \mathbb{Z}^{r_Y}$ defined by

$$(n_1, \ldots, n_{r_X}) \overset{F}{\longmapsto} \left(-\sum_{j=1}^{r_X} n_j, \sum_{j \in A_1} n_j, \ldots, \sum_{j \in A_{r_Y}} n_j\right).$$

Next, the map $H_0(p) \oplus H_0(Y) \to H_0(C_f)$ is given by

$$\left(m[p], \sum_{i=1}^{r_Y} m_i[y_i]\right) \mapsto m[p] + \sum_{i=1}^{r_Y} m_i[y_i]$$

$$= \left(m + \sum_{i=1}^{\tilde{r}_Y} m_i\right)[y_0] + \sum_{i=\tilde{r}_Y+1}^{r_Y} m_i[y_i],$$

where $y_0 \in Y$ is as above in the parenthetical comment, and where $\tilde{r}_Y$

is such that the image $f(X)$ intersects Y_i for $i = 1, \ldots, \tilde{r}_Y$ but not $i = \tilde{r}_Y + 1, \ldots, r_Y$. So, this map is the same as the map $\mathbb{Z} \oplus \mathbb{Z}^{r_Y} \xrightarrow{G} \mathbb{Z} \oplus \mathbb{Z}^{r_Y - \tilde{r}_Y}$ defined by $(m, m_1, \ldots, m_{r_Y}) \overset{G}{\mapsto} (m + m_1 + \cdots + m_{\tilde{r}_Y}, m_{\tilde{r}_Y+1}, \ldots, m_{r_Y})$.

Observe we have made the old LES look at the end like

$$\xrightarrow{\delta} \underbrace{\mathbb{Z}^{r_X}}_{\cong H_0(X)} \xrightarrow{F} \mathbb{Z} \oplus \underbrace{\mathbb{Z}^{r_Y}}_{\cong H_0(Y)} \xrightarrow{G} \mathbb{Z} \oplus \underbrace{\mathbb{Z}^{r_Y - \tilde{r}_Y}}_{\cong \tilde{H}_0(C_f)} \longrightarrow 0 \longrightarrow.$$

We augment this in the obvious way to obtain the new tail

$$\xrightarrow{\delta} \mathbb{Z}^{r_X} \xrightarrow{F'} \mathbb{Z}^{r_Y} \xrightarrow{G'} \mathbb{Z}^{r_Y - \tilde{r}_Y} \longrightarrow 0 \longrightarrow.$$

Specifically, F' is defined by $(n_1, \ldots, n_{r_X}) \overset{F'}{\mapsto} \left(\sum_{j \in A_1} n_j, \ldots, \sum_{j \in A_{r_Y}} n_j\right)$ and G' is defined by $(m_1, \ldots, m_{r_Y}) \overset{G'}{\mapsto} (m_{\tilde{r}_Y+1}, \ldots, m_{r_Y})$ (compare w/ the def's of F and G). Note: If $\tilde{r}_Y = r_Y$, then $\mathbb{Z}^{r_Y - \tilde{r}_Y} \cong \tilde{H}_0(C_f) \cong 0$ and $G' \equiv 0$.

If the resulting LES remains exact, then we are done.

<u>Exactness at $\mathbb{Z}^{r_X}$</u>: By assumption $\operatorname{im} \delta = \ker F$. So we must show $\ker F = \ker F'$. Indeed, $\ker F = \left\{(n_1, \ldots, n_{r_X}) : -\sum_{j=1}^{r_X} n_j = \sum_{j \in A_1} n_j = \cdots = \sum_{j \in A_{r_Y}} n_j = 0\right\} = \left\{(n_1, \ldots, n_{r_X}) : \sum_{j \in A_1} n_j = \cdots = \sum_{j \in A_{r_Y}} n_j\right\} = \ker F'$.

<u>Exactness at $\mathbb{Z}^{r_Y}$</u> : For concreteness, we divide into the cases $\tilde{r}_Y = r_Y$ and $\tilde{r}_Y \neq r_Y$. (i) If $\tilde{r}_Y = r_Y$, then $G' \equiv 0$ and $\ker G' = \mathbb{Z}^{r_Y}$. Recall A_i consists of those $j = 1, \ldots, r_X$ s.t. $f(X_j) \in Y_i$. If $\tilde{r}_Y = r_Y$, then all $A_i \neq \emptyset$, so F' is clearly surjective. ✓ (ii) If $\tilde{r}_Y \neq r_Y$, then some $A_i = \emptyset$ $(i = \tilde{r}_Y + 1, \ldots, r_Y)$, so F' surjects onto $\mathbb{Z} \times \cdots \times \mathbb{Z} \times \underset{\substack{\uparrow \\ \tilde{r}_Y+1}}{0} \times \cdots \times 0$, and the kernel of G' is also those $(m_1, \ldots, m_{\tilde{r}_Y}, m_{\tilde{r}_Y+1}, \ldots, m_{r_Y})$ whose last $r_Y - \tilde{r}_Y$ coordinates vanish, hence $\operatorname{im} F' = \ker G'$. ✓

<u>Exactness at $\mathbb{Z}^{r_Y - \tilde{r}_Y}$</u> : This is immediate since G' is clearly surjective. ✓

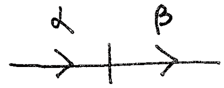
This completes the solution. $\square$.

---

(5) A <u>Lie group</u> G is a smooth manifold that is also a group s.t. the group multiplication $G \times G \to G$ and inversion $G \to G$ operations are smooth. Prove the fundamental group of a connected Lie group is abelian.

solution

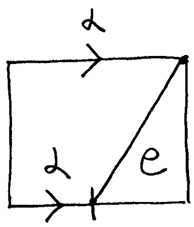
Since G is connected, we may assume our loops are based at the identity. Let $\alpha(t)$ and $\beta(t)$ be loops.

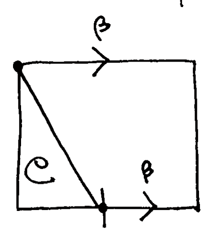
By $(\alpha * \beta)(t)$ it is meant the concatenation 

, whereas $\alpha(t) \cdot \beta(t)$ refers to the product of the paths in the group G.

We show $(\alpha * \beta)(t) \simeq \alpha(t) \cdot \beta(t) \simeq (\beta * \alpha)(t)$.

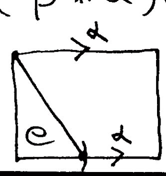
Let $F(t,s)$ and $G(t,s)$ be the path homotopies

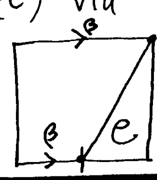
$F(t,s)$ 

$G(t,s)$ 

Then $F(t,s) \cdot G(t,s)$ is the desired path homotopy from $(\alpha * \beta)(t)$ to $\alpha(t) \cdot \beta(t)$.

And $(\beta * \alpha)(t) \simeq \alpha(t) \cdot \beta(t)$ via $F'(t,s) \cdot G'(t,s)$ where

$F'(t,s)$ 

$G'(t,s)$ 

$\square$

⑥ $M \subset \mathbb{R}^3$ embedded compact oriented surface w/out boundary of genus ≥ 1.
Prove the Gaussian curvature must vanish somewhere.

<u>solution</u>

Since M is compact embedded, it has a point of positive curvature.
Since M is of genus $g \geq 1$, and oriented, its Euler characteristic is $2-2g \leq 0$.
By Gauss-Bonnet, $\int_M K = 2\pi \chi(M) \leq 0$.
So it must have a point of ≤ 0 curvature.
By the IVT, it has a point of 0 curvature.
$\square$.