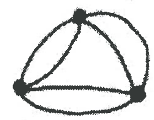
Geometry and Topology Graduate Exam
Fall 2010

Problem 1. Compute the fundamental groups of the following two graphs:

$X_1 =$  $X_2 =$ 

Problem 2. Let P_1, P_2, P_3 be three distinct points in the sphere S^2, and let X be the topological space obtained from S^2 by gluing these three points together. Compute all homology groups $H_n(X;\mathbb{Z})$.

Problem 3. Define the Gaussian (or scalar) curvature $\kappa(p)$ of an immersed surface Σ in $\mathbb{R}^3$ at the point p. Does there exist a compact immersed surface Σ without boundary in $\mathbb{R}^3$ which has $\kappa(p) = -1$ for all $p \in \Sigma$?

Problem 4. Let $M_n(\mathbb{R})$ be the set of $n \times n$ matrices with real entries. Prove that the *orthogonal group* $O(n) = \{A \in M_n(\mathbb{R}) | AA^T = \mathrm{id}\}$ is a smooth manifold. What is its dimension?

Problem 5. Let $\omega \in \Omega^{n-1}(\mathbb{R}^n - \{0\})$ be a differential form such that

$$d\omega = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$$

where $x_1, x_2, \ldots, x_n$ are the standard coordinates of $\mathbb{R}^n$. Show that, for every $p \in \mathbb{R}$, the differential form

$$\alpha = \frac{1}{(x_1^2 + x_2^2 + \cdots + x_n^2)^p}\,\omega \ \in \Omega^{n-1}(\mathbb{R}^n - \{0\})$$

is not exact. Possible hint: S^{n-1}.

Problem 6. Consider the 2-form $\omega = \sum_{i=1}^n dx_i \wedge dy_i$ on $\mathbb{R}^{2n}$ with coordinates x_1, $y_1, \ldots, x_n, y_n$. If f is a smooth function on $\mathbb{R}^{2n}$, find the vector field X such that $i_X\omega = df$, where i_X denotes the interior product. Then compute the Lie derivative $\mathcal{L}_X\omega$.

Problem 7. Let X be a topological space such that the homology group $H_p(X;\mathbb{Z})$ is finite and such that the cohomology group $H^{p+1}(X;\mathbb{Q})$ is equal to 0. Let $u \in C^{p+1}(X;\mathbb{Z}) = \mathrm{Hom}(C_{p+1}(X;\mathbb{Z}),\mathbb{Z})$ be a cochain with $du = 0$.

a. Show that, for every $\alpha \in C_p(X,\mathbb{Z})$ with $\partial\alpha = 0$, there exists $k \in \mathbb{Z} - \{0\}$ and $\beta \in C_{p+1}(X;\mathbb{Z})$ with $k\alpha = \partial\beta$.

b. Show that there exists a homomorphism

$$L_u : H_p(X;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$$

such that

$$L_u([\alpha]) = \frac{1}{k}u(\beta)$$

for every $k \in \mathbb{Z} - \{0\}$ and $\beta \in C_{p+1}(X;\mathbb{Z})$ with $k\alpha = \partial\beta$. Namely, show that $L_u([\alpha])$ is independent of k, β and of the representative α of $[\alpha] \in H_p(X;\mathbb{Z})$.

Geometry Fall 2010

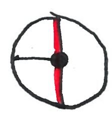
① Compute the fundamental group of the graphs

$X_1 =$  , $X_2 =$ 

<u>solution</u> Seifert – van Kampen.

Ahead of time, know answer is F_{e-v+1},

the free group on $e-v+1$ generators (also written $\mathbb{Z}^{*(e-v+1)}$).

Proof : $\pi_1(X_1) = \pi_1($  $) = \pi_1($  $) *_0 \pi_1($  $)$

$= \pi_1($  $) * \pi_1(O) = \pi_1(O)^{*3} = \mathbb{Z}^{*3}.$ ✓

$\pi_1(X_2) = \pi_1($  $) * \pi_1($  $) = \pi_1(O) * \pi_1($  $) * \pi_1($  $)$

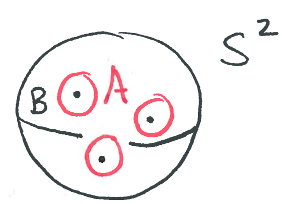
$= \pi_1(O)^{*2} * \pi_1(\infty) = \pi_1(O)^{*4} = \mathbb{Z}^{*4}.$ ✓ □

② P_1, P_2, P_3 distinct points on S^2.

$X = S^2/_{P_1 \sim P_2 \sim P_3}$. Compute the homology groups $H_n(X;\mathbb{Z})$.

<u>solution</u>

Let $A \overset{S^2}{\subset}$ consist of small disjoint disks containing P_1, P_2, P_3 respectively.



Let $B = \{P_1, P_2, P_3\}$.

Let $\tilde{X}$ be the quotient Space $S^2/P_1 \sim P_2 \sim P_3$.

Let $\pi : S^2 \to \tilde{X}$ be the natural projection.

Let $\tilde{A} := \pi(A)$ and $\tilde{B} := \pi(B)$.

Note that π gives a homeomorphism $(S^2 - B, A - B) \xrightarrow{\sim} (\tilde{X} - \tilde{A}, \tilde{A} - \tilde{B})$.

By Excision, $H_n(\tilde{X}, \tilde{A}) \cong H_n(\tilde{X} - \tilde{B}, \tilde{A} - \tilde{B}) \cong H_n(S^2 - B, A - B) \cong H_n(S^2, A)$.

we have LES in relative homology

$$\to H_{k+1}(\tilde{X}, \tilde{A}) \rightsquigarrow H_k(\tilde{A}) \to H_k(\tilde{X}) \to H_k(\tilde{X}, \tilde{A}) \rightsquigarrow H_{k-1}(\tilde{A}) \to .$$

<u>If $k > 1$:</u>

$$\underset{0}{H_k(\tilde{A})} \to H_k(\tilde{X}) \overset{\sim}{\to} H_k(\tilde{X}, \tilde{A}) \rightsquigarrow \underset{0}{H_{k-1}(\tilde{A})}$$

since $\tilde{A}$ deformation retracts to $\tilde{B}$. For the same reason $H_k(S^2) \cong H_k(S^2, A)$.

so $H_k(\tilde{X}) \cong H_k(\tilde{X}, \tilde{A}) \cong H_k(S^2, A) \cong H_k(S^2) = \begin{cases} 0, & k \neq 2 \\ \mathbb{Z}, & k = 2. \end{cases}$

If $k=1$:

$\tilde{A}$ p.c. $\qquad$ $\tilde{X}$ p.c.

$$\begin{array}{ccccccccc} H_k(\tilde{A}) & \to & H_k(\tilde{X}) & \to & H_k(\tilde{X},\tilde{A}) & \xrightarrow{\delta_1} & H_{k-1}(\tilde{A}) & \xrightarrow{H_0(i)} & H_{k-1}(\tilde{X}) \\ 0 & & & & & & \mathbb{Z} & \xrightarrow[\text{inclusion}]{} & \mathbb{Z} \\ & & & & & & x & \longmapsto & x \end{array}$$

$$H_1(\tilde{X}) \cong \ker \delta_1$$

$$\operatorname{im} \delta_1 \cong \ker H_0(i) \cong 0$$

$$\Rightarrow \quad \delta_1 = 0.$$

$$\Rightarrow \quad H_1(\tilde{X}) \cong \ker \delta_1 \cong H_1(\tilde{X},\tilde{A}) \cong H_1(S^2, A)$$

Note A is not p.c.

$$\begin{array}{ccccccccc} H_1(A) & \to & H_1(S^2) & \to & H_1(S^2,A) & \to & H_0(A) & \xrightarrow{H_0(i)} & H_0(S^2) \\ 0 & & 0 & & & & \mathbb{Z}^3 & \xrightarrow[\text{inclusion}]{} & \mathbb{Z} \\ & & & & & & (x,y,z) & \longmapsto & x+y+z \end{array}$$

$$H_1(S^2,A) \cong \ker H_0(i) \cong \mathbb{Z}^2$$ ($\operatorname{im} H_0(i) \cong \mathbb{Z}$ and one can show $(2,-1,-1), (-1,-1,2) \in \ker H_0(i)$ are linearly independent.)

(note the rank-nullity theorem holds for linear maps $R^n \xrightarrow{f} R^m$ for any PID R : $0 \to \ker f \to R^n \to \operatorname{im} f \to 0$ and since R is a PID $\operatorname{im} f$ is free, so the SES splits, so $R^n \cong \ker f \oplus \operatorname{im} f$. ✓.)

Summary : $H_0(\tilde{X}) \cong \mathbb{Z}$, $H_1(\tilde{X}) \cong \mathbb{Z}^2$, $H_2(\tilde{X}) \cong \mathbb{Z}$, $H_k(\tilde{X}) = 0$ $(k>2)$.

$\square$

(3) Define the Gaussian curvature $K(p)$ of an embedded surface Σ in $\mathbb{R}^3$ at a point $p \in \Sigma$.

Does there exist a compact embedded surface Σ w/out boundary in $\mathbb{R}^3$ s.t. $K(p) = -1$ for all $p \in \Sigma$?

This description is from Milnor in <u>Morse Theory</u>.

Let $M \xrightarrow{\vec{x}} \mathbb{R}^n$ be an embedding of a k-dim manifold M into Euclidean space.

Let $p \in M$ and let $\varphi : U \to \varphi(U) \overset{\text{open}}{\subset} \mathbb{R}^k$ be a coordinate chart around p, the coordinates written $u_1, \dots, u_k$.

Choose φ in such a way that $\left(\frac{\partial \vec{x}}{\partial u_i} \cdot \frac{\partial \vec{x}}{\partial u_j} \right)$ is the $k \times k$ identity matrix.

Let $\vec{\ell}_{ij}$ denote the projection of $\partial^2 \vec{x} / \partial u_i \partial u_j$ onto the normal line at p.

Choose a normal vector $\vec{v}$ of unit length at p.

Define the <u>principle curvatures</u> $K_1, \dots, K_k$ at p to be the eigenvalues of the matrix $(\vec{v} \cdot \vec{\ell}_{ij})$.

In the case $n = 3, k = 2, M \equiv \Sigma$, the <u>Gaussian curvature</u> $K(p)$ at the point $p \in \Sigma$ is defined by $K(p) := K_1(p) K_2(p)$.

A theorem, first discovered by Gauss, is that $K(p)$ does not depend on the choices we made (including the choice of embedding itself).

Regarding the second part, after searching a bit, it appears the answer is positive if we allow the embedding to be C^1 (see the Nash-Kuiper embedding theorem) and negative if we require the embedding be C^∞ ← see one of Hilbert's theorems, perhaps. (Not 100% sure though). $\square$

(4) Prove $O(n) = \{ A \in M_n(\mathbb{R}) : AA^T = I \}$ is a manifold. What is its dimension?

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solution

Define a map $M_n(\mathbb{R}) \xrightarrow{f} S_n(\mathbb{R}) \equiv \{\text{symmetric matrices in } M_n(\mathbb{R})\}$

by $A \overset{f}{\mapsto} AA^T$. We calculate $T_A f : M_n(\mathbb{R}) \to S_n(\mathbb{R})$.

$$T_A f(B) = \frac{d}{dt}\Big|_{t=0} f(A+tB)$$

$$= \lim_{h\to 0} \frac{(A+hB)(A+hB)^T - AA^T}{h}$$

$$= \lim_{h\to 0} \frac{hAB^T + hBA^T + h^2 BB^T}{h} = AB^T + BA^T.$$

If $A \in O(n)$ and $C \in S_n(\mathbb{R})$

we want to find $B \in M_n(\mathbb{R})$ s.t. $AB^T + BA^T = C$.

Take $B = CA/2$:

$$AB^T + BA^T = \frac{AA^TC^T}{2} + \frac{CAA^T}{2} = \frac{C}{2} + \frac{C}{2} = C.$$

By the regular value theorem, $O(n)$ is a submanifold. Its dimension is $n^2 - n(n+1)/2 = n(n-1)/2$. $\square$

(5) Let $\omega \in \Omega^{n-1}(\mathbb{R}^n - \{0\})$ s.t.

$$d\omega = dX_1 \wedge dX_2 \wedge \cdots \wedge dX_n$$

where $X_1, X_2, \ldots, X_n$ are the standard coordinates of $\mathbb{R}^n$.

Show for every $p \in \mathbb{R}$ that the differential form

$$\alpha = \frac{1}{(X_1^2 + X_2^2 + \cdots + X_n^2)^p} \omega \in \Omega^{n-1}(\mathbb{R}^n - \{0\})$$

is not exact. Hint: S^{n-1}.

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solution

If α is exact, then $d\alpha = 0$.

$$\int_{S^{n-1}} \alpha = \int_{D^n - \{0\}} d\alpha = 0$$

where $D^n = \{(X_1, \ldots, X_n) \in \mathbb{R}^n : X_1^2 + \cdots + X_n^2 \leq 1\}$.

On S^{n-1}, $\alpha = \omega$.

$$\int_{S^{n-1}} \alpha = \int_{S^{n-1}} \omega = \int_{D^n - \{0\}} dX_1 \wedge dX_2 \wedge \cdots \wedge dX_n \neq 0. \quad \Rightarrow\Leftarrow.$$

Therefore α cannot be closed. $\square$

⑥ Consider the 2-form $\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$

on $\mathbb{R}^{2n}$ w/ coordinates $x_1, y_1, \dots, x_n, y_n$.

Let f be a smooth function on $\mathbb{R}^{2n}$.

Find the vector field X s.t. $i_X \omega = df$,

where i_X denotes the interior product.

Compute the Lie derivative $\mathcal{L}_X \omega$.

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solution $\quad i_X : \Omega^p(M) \longrightarrow \Omega^{p-1}(M)$ is defined by

$$(i_X \omega)_x (v_1, \dots, v_{p-1}) = \omega_x(X_x, v_1, \dots, v_{p-1}).$$

Recall if $\alpha \in \Omega^p(M)$ and $\beta \in \Omega^q(M)$ then

$$(\alpha \wedge \beta)_x(v_1, \dots, v_{p+q}) = \frac{1}{p!\,q!} \sum_{\sigma \in S_{p+q}} \operatorname{sign}(\sigma)\, \alpha_x(v_{\sigma(1)}, \dots, v_{\sigma(p)})\, \beta_x(v_{\sigma(p+1)}, \dots, v_{\sigma(p+q)}).$$

So for $\omega \in \Omega^2(\mathbb{R}^{2n})$ as above, for any v.f. X

$$(i_X \omega)_x(v) = \sum_{i=1}^{n} dx_i(X_x)\, dy_i(v) - dx_i(v)\, dy_i(X_x)$$

$$= \sum_{i=1}^{n} X_x^{2i-1} v^{2i} - v^{2i-1} X_x^{2i}.$$

We want to expand this in the basis $\{dx_i, dy_i\}$ of $\Omega^1(\mathbb{R}^{2n})$.

The dx_j coordinate is obtained by taking $v = e_{2j-1}$ and the dy_j coordinate $v = e_{2j}$.

So the former is $\sum_{i=1}^{n} X_x^{2i-1} e^{2i}_{2j-1} - e^{2i-1}_{2j-1} X_x^{2i}$

$= -X_x^{2j}$, and the latter is X_x^{2j-1}.

So in summary $(i_X \omega)_x = \sum_{j=1}^{n} (-X_x^{2j}) dx_j + X_x^{2j-1} dy_j$.

If $i_X \omega = df$ then we require $X^{2j} = -\frac{\partial f}{\partial x_j}$, $X^{2j-1} = \frac{\partial f}{\partial y_j}$.

Cartan's formula is $L_X \omega = i_X(d\omega) + d(i_X \omega)$.

Note $d\omega = 0$. Hence, $L_X \omega = d(i_X \omega) = d^2 f = 0$.

$\square$

(7) Let X be a space s.t. $H_p(X;\mathbb{Z})$ is finite, and s.t. $H^{p+1}(X;\mathbb{Q})=0$. Let $u \in C^{p+1}(X;\mathbb{Z}) \equiv \mathrm{Hom}_{\mathbb{Z}}(C_{p+1}(X;\mathbb{Z}),\mathbb{Z})$ satisfy $du=0$. Here $d: C^{p+1}(X;\mathbb{Z}) \to C^{p+2}(X;\mathbb{Z})$ is the coboundary map.

(a) Show for all $\alpha \in C_p(X;\mathbb{Z})$ w/ $\partial\alpha = 0$ that there exists $k \in \mathbb{Z}-\{0\}$ and $\beta \in C_{p+1}(X;\mathbb{Z})$ s.t. $k\alpha = \partial\beta$.

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solution to (a)

Since $H_p(X,\mathbb{Z})$ is a finite abelian group, and since $\partial\alpha = 0$, we have

$$[\alpha] \in H_p(X,\mathbb{Z}) \cong \mathbb{Z}_{d_1} \oplus \mathbb{Z}_{d_2} \oplus \cdots \oplus \mathbb{Z}_{d_n}.$$

$d_1 | d_2 | \cdots | d_n$, $d_i > 1$

So $\underset{\overset{\Vert}{[d_n\alpha]}}{d_n[\alpha]} = 0 \in H_p(X,\mathbb{Z})$

i.e. there is $\beta \in C_{p+1}(X;\mathbb{Z})$ s.t. $\partial\beta = d_n\alpha$; set $k = d_n$.

(b) Show there exists a $\mathbb{Z}$-map

$$L_u : H_p(X;\mathbb{Z}) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

satisfying $L_u([\alpha]) = \frac{1}{k} u(\beta)$

for every $k \in \mathbb{Z} - \{0\}$ and $\beta \in C_{p+1}(X;\mathbb{Z})$

w/ $\partial\beta = k\alpha$. Namely, check $L_u([\alpha])$

is independent of k, β, and the representative α.

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solution to (b)

we will need to use $H^{p+1}(X;\mathbb{Q}) = 0$,

which translates to: if $u \in C^{p+1}(X;\mathbb{Q}) \equiv \mathrm{Hom}_{\mathbb{Q}}(C_{p+1}(X;\mathbb{Q}),\mathbb{Q})$

satisfies $du = 0$, then there is $w \in C^p(X;\mathbb{Q}) \equiv \mathrm{Hom}_{\mathbb{Q}}(C_p(X;\mathbb{Q}),\mathbb{Q})$

s.t. $u = dw$.

We are given a $\mathbb{Z}$-map $u : \underset{\mathbb{Z}^J}{\overset{}{C_{p+1}(X;\mathbb{Z})}} \to \mathbb{Z}$

where $J = \{\text{maps } \Delta^{p+1} \to X\}$. The inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$

induces a map $u : C_{p+1}(X;\mathbb{Q}) \xrightarrow{\cong \mathbb{Q}^J} \mathbb{Q}$ defined by the same formula.

Recall $d : C^{p+1}(X;\mathbb{Q}) \to C^{p+2}(X;\mathbb{Q})$ is defined on $u \in C^{p+1}(X;\mathbb{Q})$ and $(p+2)$-simplices σ by

$$du(\sigma) = \underbrace{(-1)^{p+2}}_{\text{used in Milnor, but seems outdated (not in Hatcher or by Bonahon).}} u(\partial\sigma) = (-1)^{p+2} \sum_{i=0}^{p+2} (-1)^i u(\sigma \circ f_i)$$

where $f_i : \Delta^{p+1} \to \Delta^{p+2}$ is the i^{th} face map.

For our u from above, we are assuming $u \in C^{p+1}(X;\mathbb{Z})$ vanishes under $d : C^{p+1}(X;\mathbb{Z}) \to C^{p+2}(X;\mathbb{Z})$. It is clear then $u \in C^{p+1}(X;\mathbb{Q})$ vanishes under $d : C^{p+1}(X;\mathbb{Q}) \to C^{p+2}(X;\mathbb{Q})$, by the above formula since $u(\sigma \circ f_i) = 0$.

So there exists $w \in C^p(X;\mathbb{Q})$ s.t. $dw = u$.

We turn to the $\mathbb{Z}$-map $L_u : H_p(X;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$. By part (a), for every $[\alpha] \in H_p(X;\mathbb{Z})$ there is $k \in \mathbb{Z} - \{0\}$ and $\beta \in C_{p+1}(X;\mathbb{Z})$ s.t. $k\alpha = \partial\beta$. Fix the representative α. Assume (k', β') also satisfies $k'\alpha = \partial\beta'$. We need to show $\frac{1}{k}u(\beta) - \frac{1}{k'}u(\beta') \in \mathbb{Z}$.

In fact, we show this integer is 0.

Indeed, $\frac{1}{k}u(\beta) - \frac{1}{k'}u(\beta') = u\left(\frac{\beta}{k} - \frac{\beta'}{k'}\right)$

$$= d\omega\left(\frac{\beta}{k} - \frac{\beta'}{k'}\right) = (-1)^{p+1}\omega\left(\frac{\partial\beta}{k} - \frac{\partial\beta'}{k'}\right)$$

$$= (-1)^{p+1}\,\omega\left(\frac{k\alpha}{k} - \frac{k'\alpha}{k'}\right) = 0.$$

It remains to show that if $\alpha \in C_p(X;\mathbb{Z})$ is exact, then $\frac{1}{k}u(\beta) \in \mathbb{Z}$.

~~Indeed,~~ $\cancel{\frac{1}{k}u(\beta) = d\omega(\beta/k) = \omega\left(\frac{\partial\beta}{k}\right)}$

$\cancel{= \omega(\alpha)}$

Since α is exact, $\alpha = \partial\beta$, i.e. $k=1$.

Then $\frac{1}{k}u(\beta) = u(\beta) \in \mathbb{Z}$

since $u \in C^{p+1}(X;\mathbb{Z})$. $\square$