Intro

Here are my solutions to some of USC's qualifying exams. A lot of the solutions here are ones I came up with myself, but many other ones are adapted from ideas that I found either online or in textbooks, so I definitely don't claim all of the credit for everything here (I did make the diagrams, however). For some problems, I've given some background information in purple; you don't need to know it to solve the problem, but it may be interesting. I've put a question mark (?) next to solutions I didn't feel completely confident in; and although I've done my best to avoid this, some of the other solutions may contain mistakes too, so please keep that in mind. Thanks and good luck! – Alec.

Notation

Below is a guide of notation and terminology you'll find throughout my solutions. If a problem uses the symbols below to mean something else, then I'll do the same for that problem.

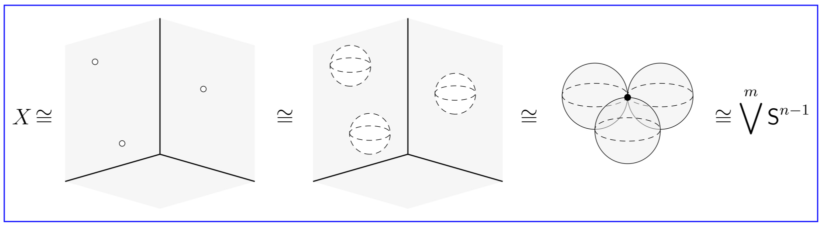
- ◦ In the context of geometry, all manifolds, functions, etc. are smooth unless otherwise stated.
- ◦ In the context of topology, all functions are continuous unless otherwise stated.
- Σ_n denotes the symmetric group on $n \in \mathbb{N}$ symbols.
- B^n denotes the closed (unit) n-ball, and $\mathsf{S}^n = \partial\mathsf{B}^{n+1}$ the (unit) n-sphere, for $n \in \mathbb{N}$.
- $\mathsf{C}^\infty(M)$ denotes the $\mathbb{R}$-algebra of (smooth) functions from a manifold M to $\mathbb{R}$.
- F_n denotes the free group with n generators, for $n \in \mathbb{N}$.
- $\mathsf{Mat}_n(R)$ denotes the ring of $n \times n$ matrices with entries in a ring R, for $n \in \mathbb{N}$.
- $\mathfrak{X}(M)$ denotes the space of (smooth) vector fields on a manifold M.
- $\mathbb{Z}_n$ denotes the ring of integers modulo an integer $n \in \mathbb{N}$.

Exams

2005, Fall

Problem 1.

Let $n \geq 3$ and set $X := \mathbb{R}^n \setminus \{x_1, \ldots, x_m\}$, with $x_i \neq x_j$ if $i \neq j$. Expanding each missing point into a small bubble, we may shrink the complement of these bubbles to a point to obtain a wedge of m copies of S^{n-1}.



Thus $\pi_1(X) \cong \pi_1\left(\bigvee^m \mathsf{S}^{n-1}\right) \cong \pi_1(\mathsf{S}^{n-1})^{*m} \cong 1.$ □

Problem 2.

Equivalence classes of connected covers of $\mathbb{R}\mathsf{P}^2 \times \mathbb{R}\mathsf{P}^2$ are in bijection with the 4 subgroups of

$$\pi_1(\mathbb{R}\mathsf{P}^2 \times \mathbb{R}\mathsf{P}^2) \cong \pi_1(\mathbb{R}\mathsf{P}^2) \times \pi_1(\mathbb{R}\mathsf{P}^2) \cong \mathbb{Z}_2^{\oplus 2}.$$

The identity subgroup corresponds to the universal cover $\mathsf{S}^2 \times \mathsf{S}^2$; the entire group corresponds to the trivial cover $\mathbb{R}\mathsf{P}^2 \times \mathbb{R}\mathsf{P}^2$; the subgroups generated by $(0,1)$ and $(1,0)$ correspond to the covers $\mathsf{S}^2 \times \mathbb{R}\mathsf{P}^2$ and $\mathbb{R}\mathsf{P}^2 \times \mathsf{S}^2$, respectively. □

Problem 3.

Assume $(\alpha \wedge \alpha)_x \neq 0$ for all $x \in \mathsf{S}^4$. Then $\alpha \wedge \alpha$ is a volume form on S^4, so $\int_{\mathsf{S}^4} \alpha \wedge \alpha \neq 0$. But

$$\int_{\mathsf{S}^4} \alpha \wedge \alpha = \int_{\mathsf{B}^5} \mathsf{d}(\alpha \wedge \alpha) = \int_{\mathsf{B}^5} [\underbrace{(\mathsf{d}\alpha)}_{=0} \wedge \alpha + \alpha \wedge \underbrace{(\mathsf{d}\alpha)}_{=0}] = 0,$$

by Stokes, a contradiction. □

Problem 4 (?)**.**

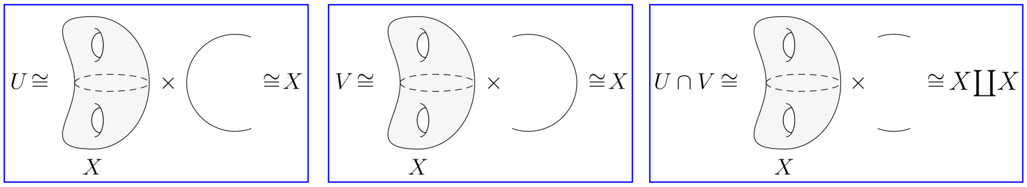
The Euler characteristic of M is that of a genus-3 surface, $\chi(M) = -4$. And, ∂M is the boundary of the plane, so integrating the geodesic curvature k_{g} over this boundary yields the sum of its exterior angles, namely $4(3\pi/2) = 6\pi$. Then

$$\iint_M K \mathsf{d}A = 2\pi\chi(M) - \int_{\partial M} k_{\mathsf{g}} \mathsf{d}s = 2\pi(-4) - 6\pi = -14\pi$$

by Gauss-Bonnet. □

Problem 5 (?)**.**

Defining



gives $U \cup V \cong X \times \mathsf{S}^1$. For any $j \in \mathbb{Z}$, by Mayer-Vietoris we have an exact sequence

$$\mathsf{H}_j(X)^{\oplus 2} \xrightarrow{\iota_j} \mathsf{H}_j(X)^{\oplus 2} \xrightarrow{f} \mathsf{H}_j(X \times \mathsf{S}^1) \xrightarrow{\partial} \mathsf{H}_{j-1}(X)^{\oplus 2} \xrightarrow{\iota_{j-1}} \mathsf{H}_{j-1}(X)^{\oplus 2}.$$

Here, $\iota_j = ((\iota_U)_*, (\iota_V)_*)$ is the map induced by the inclusions $\iota_U : U \cap V \hookrightarrow U$ and $\iota_V : U \cap V \hookrightarrow V$, and acts as $\iota_j([\omega]) = ([\omega], [\omega])$ on any $[\omega] \in \mathsf{H}_j(X \coprod X) \cong \mathsf{H}_j(X)^{\oplus 2}$. So $\mathsf{im}(\iota_j) \cong \mathsf{H}_j(X)$ and consequently $\mathsf{ker}(\iota_j) \cong \mathsf{H}_j(X)$ also. We have a similar result for $j-1$. Then

$$\mathsf{im}(\partial) \cong \mathsf{ker}(\iota_{j-1}) \cong \mathsf{H}_{j-1}(X), \quad \mathsf{ker}(\partial) \cong \mathsf{im}(f) \cong \mathsf{H}_j(X)$$

since $\mathsf{ker}(f) \cong \mathsf{im}(\iota_j) \cong \mathsf{H}_j(X)$, and therefore $\mathsf{H}_j(X \times \mathsf{S}^1) \cong \mathsf{H}_j(X) \oplus \mathsf{H}_{j-1}(X)$. □

Problem 6.

- Let M be an m-manifold. Via the canonical projection $\pi : \mathsf{T}^*M \twoheadrightarrow M$, we may lift any chart (U, φ) of M to a subset $\mathsf{T}^*U := \pi^{-1}(U) \subset \mathsf{T}^*M$ and a map $\varphi \times \mathsf{d}\varphi : \mathsf{T}^*U \to \varphi(U) \times \mathbb{R}^m$. Requiring this map to be a homeomorphism defines a topology on T^*M and makes $(\mathsf{T}^*U, \varphi \times \mathsf{d}\varphi)$ into a chart for T^*M. This topology inherits the second countable and Hausdorff properties from that of M. Moreover, given any two charts $(\mathsf{T}^*U_1, \varphi_1 \times \mathsf{d}\varphi_1), (\mathsf{T}^*U_2, \varphi_2 \times \mathsf{d}\varphi_2)$ with $\mathsf{T}^*U_1 \cap \mathsf{T}^*U_2 \neq \varnothing$, we have $U_1 \cap U_2 \neq \varnothing$ and so the transition map

$$\tau := (\varphi_2 \times \mathsf{d}\varphi_2) \circ (\varphi_1 \times \mathsf{d}\varphi_1)^{-1} = (\varphi_2 \circ \varphi_1^{-1}) \times (\mathsf{d}\varphi_2 \circ \mathsf{d}\varphi_1^{-1})$$

  is smooth since its first component is a transition map of M and its second component is linear. Therefore T^*M is a (smooth) manifold.

- It remains to check that any transition map τ as above is orientation-preserving. Say T^*U_1 has local coordinates $(x_1, \ldots, x_m, v_1, \ldots, v_m)$, and express the differential $\mathsf{d}\tau_{(x,v)}$ at a point $(x, v) \in \mathsf{T}^*U_1$ as a $(2m) \times (2m)$ block matrix.

  (i) In the upper-left block, we differentiate the first m entries of τ w.r.t. x and obtain the usual Jacobian $\mathsf{d}(\varphi_2 \circ \varphi_1^{-1})_x$.

  (ii) In the upper-right block, we differentiate the first m entries of τ, which are independent of v, w.r.t. v, and obtain a 0 block.

  (iii) In the lower-right block, we differentiate the last m entries of τ w.r.t. v. For $1 \leq i, j \leq m$, the ij-entry in this block is

$$\partial_{v_j}(\mathsf{d}\varphi_2 \circ \mathsf{d}\varphi_1^{-1})_i = \partial_{v_j} \sum_{k=1}^{m} \partial_{x_k}(\varphi_2 \circ \varphi_1^{-1})_i(x) v_k = \partial_{x_j}(\varphi_2 \circ \varphi_1^{-1})_i(x),$$

  which coincides with the ij-entry of the Jacobian $\mathsf{d}(\varphi_2 \circ \varphi_1^{-1})_x$.

Hence

$$\mathsf{det}(\mathsf{d}\tau_{(x,v)}) = \begin{vmatrix} \mathsf{d}(\varphi_2 \circ \varphi_1^{-1})_x & 0 \\ * & \mathsf{d}(\varphi_2 \circ \varphi_1^{-1})_x \end{vmatrix} = \mathsf{det}\left(\mathsf{d}(\varphi_2 \circ \varphi_1^{-1})_x\right)^2 > 0$$

as desired.

□

Problem 7.

Background. This problem concerns *compactly supported de Rham cohomology.* Given a manifold M^m and $0 \leq k \leq m$, we use this cohomology to define the *cup product*,

$$\smile: \mathsf{H}^k(M) \times \mathsf{H}_\mathsf{c}^{m-k}(M) \to \mathbb{R},$$

given by $[\alpha] \smile [\beta] := \int_M \alpha \wedge \beta$. Poincaré duality then states that the map $\mathsf{H}^k(M) \to \mathsf{H}_\mathsf{c}^{m-k}(M)^*$ given by $[\alpha] \mapsto [\alpha] \smile (\cdot)$ is an isomorphism.

Denote by $\mathsf{H}_\mathsf{c}^\bullet(\mathbb{R})$ the cohomology of $(\Omega_\mathsf{c}^\bullet(\mathbb{R}), \mathsf{d}^\bullet)$. For all $j \geq 2$ we clearly have $\mathsf{H}_\mathsf{c}^j(\mathbb{R}) \cong 0$ since $\Omega_\mathsf{c}^j(\mathbb{R}) \cong 0$, so it remains to compute $\mathsf{H}_\mathsf{c}^j(\mathbb{R})$ for $j = 0, 1$. Observe that both $\Omega_\mathsf{c}^0(\mathbb{R})$ and $\Omega_\mathsf{c}^1(\mathbb{R})$ are canonically isomorphic to $\mathsf{C}_\mathsf{c}^\infty(\mathbb{R})$, the subset of $\mathsf{C}^\infty(\mathbb{R})$ consisting of compactly supported functions.

- Note that $f \in \mathsf{H}_\mathsf{c}^0(\mathbb{R}) \cong \mathsf{ker}(\mathsf{d}^0)$ if and only if f is constant, whereby $f \equiv 0$ since no other such function is compactly supported. Thus $\mathsf{H}_\mathsf{c}^0(\mathbb{R}) \cong 0$.
- Consider the map $I : \mathsf{H}_\mathsf{c}^1(\mathbb{R}) \cong \mathsf{C}_\mathsf{c}^\infty(\mathbb{R}) \to \mathbb{R}$ given by $I(f) := \int_\mathbb{R} f$. Note that $\mathsf{ker}(I) \subset \mathsf{im}(\mathsf{d}^0)$ since if $f \in \mathsf{ker}(I)$, then $f \cong \mathsf{d}g$ where g is the compactly supported function given by $g(x) := \int_{-\infty}^x f(x)\mathsf{d}x$. Conversely if $f \in \mathsf{im}(\mathsf{d}^0)$, then $f = \mathsf{d}g$ for some $g \in \mathsf{C}_\mathsf{c}^\infty(\mathbb{R})$, and

$$\int_\mathbb{R} f = \int_{-\infty}^{\infty} g'(x)\mathsf{d}x = \lim_{x\to\infty} g(x) - \lim_{x\to-\infty} g(x) = 0,$$

  so this shows $\mathsf{im}(\mathsf{d}^0) \cong \mathsf{ker}(I)$. Then $\mathsf{H}_\mathsf{c}^1(\mathbb{R}) \cong \mathsf{ker}(\mathsf{d}^1)/\mathsf{im}(\mathsf{d}^0) \cong \mathsf{C}_\mathsf{c}^\infty(\mathbb{R})/\mathsf{ker}(I) \cong \mathbb{R}$, since I is clearly surjective.

In summary, $\mathsf{H}_\mathsf{c}^j(\mathbb{R}) \cong \begin{cases} \mathbb{R} & j = 1, \\ 0 & \text{else.} \end{cases}$ □

2006, Spring

Problem 1.

Let $f : \mathbb{R}^4 \to \mathbb{R}$ be given by $f(x, y, z, w) := x^2 + xy^3 + yz^4 - w^5 + 1$. To show that $X := f^{-1}(0) \subset \mathbb{R}^4$ is a manifold, it's enough to show that the linear map

$$\mathsf{d}f_{(x,y,z,w)} = \begin{pmatrix} 2x + y^3 & 3xy^2 + z^4 & 4yz^3 & -5w^4 \end{pmatrix}$$

from X to $\mathbb{R}$ is surjective for all $(x, y, z, w) \in X$; so we need at least one entry in this matrix to be nonzero. To see this, let $(x, y, z, w) \in X$ and observe that at least one coordinate is nonzero by definition of X.

- Say $x \neq 0$. If $y = 0$ then $2x + y^3 \neq 0$. If $y \neq 0$ and $z = 0$ then $3xy^2 + z^4 \neq 0$. If $y, z \neq 0$ then $4yz^3 \neq 0$.
- Say $y \neq 0$. If $z \neq 0$ then $4yz^3 \neq 0$. If $z = 0$ and $x \neq 0$ then $3xy^2 + z^4 \neq 0$. If $z, x = 0$ then $2x + y^3 \neq 0$.
- Say $z \neq 0$. If $y \neq 0$ then $4yz^3 \neq 0$. If $y = 0$ then $3xy^2 + z^4 \neq 0$.
- Say $w \neq 0$. Then $-5w^4 \neq 0$.

$\square$

Problem 2.

(a) Given a manifold X, the *de Rham cochain complex* $(\Omega^\bullet(X), \mathsf{d}^\bullet)$ is defined in each degree $j \in \mathbb{Z}$ by $\Omega^j(X) := \{\omega \text{ a smooth } j\text{-form on } X\}$ and $\mathsf{d}^j : \Omega^j(X) \to \Omega^{j+1}(X)$ the usual exterior differential. The *j-th de Rham cohomology group* of X is the quotient $\mathsf{H}^j_{\mathsf{dR}}(X) := \mathsf{ker}(\mathsf{d}^j)/\mathsf{im}(\mathsf{d}^{j-1})$.

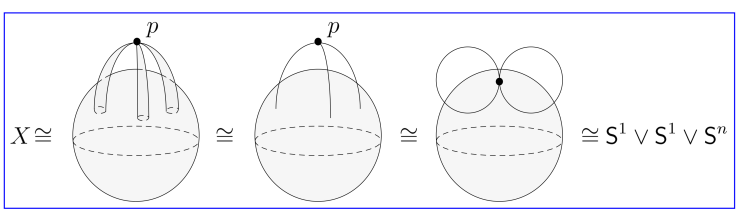
(b) Firstly, $\mathsf{H}^j_{\mathsf{dR}}(\mathbb{R}) \cong 0$ for any $j \geq 2$ since $\Omega^j(\mathbb{R}) = 0$ in this case. Now note that both $\Omega^0(\mathbb{R})$ and $\Omega^1(\mathbb{R})$ are canonically isomorphic to $\mathsf{C}^\infty(\mathbb{R})$. Then

$$\mathsf{H}^0_{\mathsf{dR}}(\mathbb{R}) \cong \mathsf{ker}(\mathsf{d}^0) \cong \{f \in \mathsf{C}^\infty(\mathbb{R}) \mid \mathsf{d}f = 0\} \cong \{f \in \mathsf{C}^\infty(\mathbb{R}) \mid f \text{ a constant}\} \cong \mathbb{R}.$$

Moreover, any $f \in \Omega^1(\mathbb{R})$ may be written as $f = \mathsf{d}g$ for $g \in \Omega^0(\mathbb{R})$ given by $g(x) := \int_{-\infty}^x f(x)\mathsf{d}x$, and so $\mathsf{im}(\mathsf{d}^0) = \Omega^1(\mathbb{R})$. Thus $\mathsf{H}^1_{\mathsf{dR}}(\mathbb{R}) \cong \mathsf{ker}(\mathsf{d}^1)/\mathsf{im}(\mathsf{d}^0) \cong \Omega^1(\mathbb{R})/\Omega^1(\mathbb{R}) \cong 0.$ $\square$

Problem 3.

By pinching the points q, r, s together and then transforming the shape as shown, we obtain a wedge of S^n with two copies of S^1.



Hence by van Kampen, $\pi_1(X) \cong \pi_1(\mathsf{S}^1) * \pi_1(\mathsf{S}^1) * \pi_1(\mathsf{S}^n) \cong \begin{cases} \mathsf{F}_3 & n = 1, \\ \mathsf{F}_2 & n \geq 2, \end{cases}.$ $\square$

Problem 4.

The canonical volume form on $\mathbb{R}^4$ with coordinates (x, y, z, w) is $\mathsf{d}x \wedge \mathsf{d}y \wedge \mathsf{d}z \wedge \mathsf{d}w$. Hence

$$\int_{\mathsf{S}^3} \omega = \int_{\mathsf{B}^4} \mathsf{d}\omega = \int_{\mathsf{B}^4} \mathsf{d}w \wedge \mathsf{d}x \wedge \mathsf{d}y \wedge \mathsf{d}z = -\int_{\mathsf{B}^4} \mathsf{d}x \wedge \mathsf{d}y \wedge \mathsf{d}z \wedge \mathsf{d}w = -\mathsf{vol}(\mathsf{B}^4)$$

by Stokes. □

Problem 5.

Background. The pairing $\smile : \mathsf{H}^1_{\mathsf{dR}}(T) \otimes \mathsf{H}^1_{\mathsf{dR}}(T) \to \mathbb{R}$ referenced in this problem is the *cup product* discussed in problem 7 of 2005, Fall.

Since $\frac{1}{2}\mathsf{dim}_{\mathbb{R}}(\mathsf{H}^1_{\mathsf{dR}}(S)) = \mathsf{g}(S) < \mathsf{g}(T) = \frac{1}{2}\mathsf{dim}_{\mathbb{R}}(\mathsf{H}^1_{\mathsf{dR}}(T))$, the map $h^* : \mathsf{H}^1_{\mathsf{dR}}(T) \to \mathsf{H}^1_{\mathsf{dR}}(S)$ is has nontrivial kernel. So, suppose $\alpha \in \mathsf{ker}(h^*)$ is nonzero. Then the map

$$\alpha \smile (\cdot) : \mathsf{H}^1_{\mathsf{dR}}(T) \to \mathsf{Hom}_{\mathbb{R}}(\mathsf{H}^1_{\mathsf{dR}}(T), \mathbb{R})$$

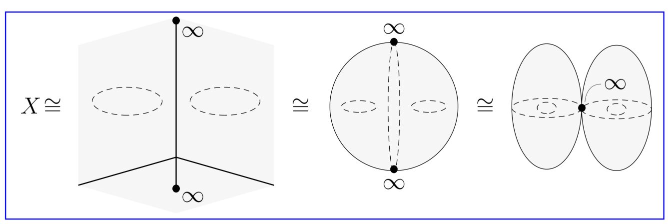
given by $\eta \mapsto \alpha \smile \eta := \int_T \alpha \wedge \eta$ is nonzero, since the pairing $\smile : \mathsf{H}^1_{\mathsf{dR}}(T) \otimes \mathsf{H}^1_{\mathsf{dR}}(T) \to \mathbb{R}$ given by $\omega \smile \eta := \int_T \omega \wedge \eta$ is nondegenerate. Thus there's some element $\beta \in \mathsf{H}^1_{\mathsf{dR}}(T)$ such that $\int_T \alpha \wedge \beta \neq 0$, and so

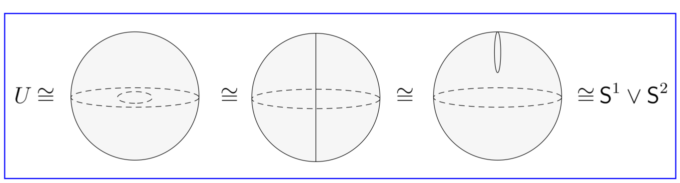
$$\mathsf{deg}(h) \underbrace{\int_T \alpha \wedge \beta}_{\neq 0} = \int_S h^*(\alpha \wedge \beta) = \int_S \underbrace{(h^*\alpha)}_{=0} \wedge (h^*\beta) = 0.$$

□

Problem 6.

- Let X be the complement of the unlink in S^3. By the homotopy below, we view X as a wedge sum of two copies of U, where U is a solid sphere with a circle removed inside. In U, we first stretch the missing circle until we're left with the surface of the sphere together with a line segment connecting the poles; we then translate the south pole along the surface and onto the north pole to obtain the wedge sum shown.



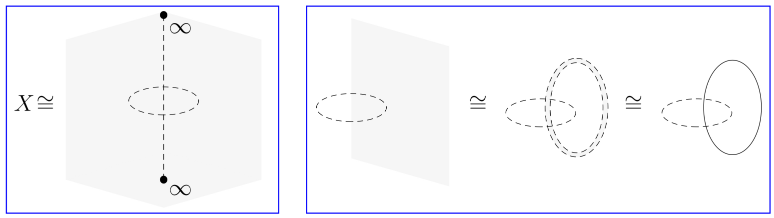


Hence $X \cong U \vee U \cong \mathsf{S}^1 \vee \mathsf{S}^1 \vee \mathsf{S}^2 \vee \mathsf{S}^2$, and so

$$\mathsf{H}_j(X) \cong \mathsf{H}_j(\mathsf{S}^1)^{\oplus 2} \oplus \mathsf{H}_j(\mathsf{S}^2)^{\oplus 2} \cong \begin{cases} \mathbb{Z} & j = 0, \\ \mathbb{Z}^{\oplus 2} & j = 1, 2, \\ 0 & \text{else}, \end{cases}$$

where the $j = 0$ case follows from the fact that X is path connected. $\square$

- Let X be the complement of the Hopf link in S^3. We assume w.l.o.g. that one of the circles passes through ∞, and hence is visualized as a vertical axis in $\mathbb{R}^3$, surrounded by the second circle. Then X is the union of all vertical planes starting at this axis, and each such plane is equivalent to a circle itself, as shown.

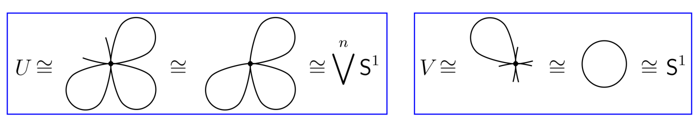


It follows that $X \cong \mathsf{S}^1 \times \mathsf{S}^1 \cong \mathsf{T}^2$, and $\mathsf{H}_j(X) \cong \mathsf{H}_j(\mathsf{T}^2) \cong \begin{cases} \mathbb{Z} & j = 0, \\ \mathbb{Z}^{\oplus 2} & j = 1, \\ \mathbb{Z} & j = 2, \\ 0 & \text{else.} \end{cases}$ $\square$

Problem 7.

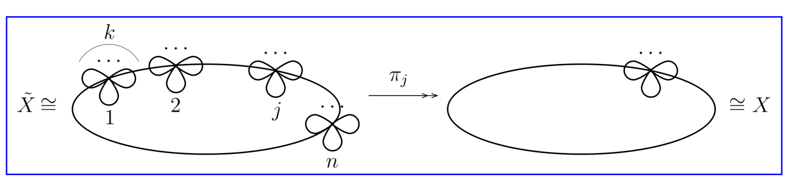
Background. In part (b) we prove the Nielsen-Schreier theorem.

(a) We proceed by induction on n. The case $n = 0$ is immediate since $\pi_1(\mathsf{S}^1) \cong \mathbb{Z} \cong \mathsf{F}_1$, so let $n \geq 1$ be arbitrary and assume $\pi_1\left(\bigvee^n \mathsf{S}^1\right) \cong \mathsf{F}_n$. Defining



gives $U \cup V \cong \bigvee^{n+1} \mathsf{S}^1$ and $U \cap V \cong *$, so $\pi_1\left(\bigvee^{n+1} \mathsf{S}^1\right) \cong \pi_1\left(\bigvee^n \mathsf{S}^1\right) * \pi_1(\mathsf{S}^1) \cong \mathsf{F}_n * \mathsf{F}_1 \cong \mathsf{F}_{n+1}$ by van Kampen. $\square$

(b) Let $X := \bigvee^{n+1} \mathsf{S}^1$. If $H \subset \mathsf{F}_{n+1} \cong \pi_1(X)$ is a subgroup with $[\mathsf{F}_{n+1} : H] = k$, then $H \cong \pi_1(\tilde{X})$ for some k-fold covering space $\tilde{X} \twoheadrightarrow X$. Note that $\tilde{X}$ is a connected graph since it's a covering space of a connected graph, and thus $\tilde{X}$ is homotopy equivalent to a wedge of circles. We observe that the covering space



obtained by attaching k copies of $\bigvee^n \mathsf{S}^1$ to a base circle gives the desired wedge product, and

$$H \cong \pi_1(\tilde{X}) \cong \pi_1\Big(\bigvee^{kn+1} \mathsf{S}^1\Big) \cong \mathsf{F}_{kn+1}.$$

$\square$

2006, Fall

Problem 1.

Assume f is nonsurjective; then $\mathsf{deg}(f) = 0$. The map $\int_M : \mathsf{H}^n_{\mathsf{dR}}(M) \to \mathbb{R}$ is an isomorphism and the map $f^* : \mathsf{H}^n_{\mathsf{dR}}(N) \to \mathsf{H}^n_{\mathsf{dR}}(M)$ is surjective, so $\int_M f^* : \mathsf{H}^n_{\mathsf{dR}}(N) \to \mathbb{R}$ is surjective. But by definition of degree, $\int_M f^* = \mathsf{deg}(f) \int_N = 0$, and the zero map is nonsurjective. $\square$

Problem 2.

- We first set $X_p := (\mathsf{T}^2 \coprod D_1)/\sim$, where we identify each point $e^{i\theta} \in \partial D_1, 0 \leq \theta < 2\pi$, with the point $(e^{ip\theta}, 1) \in \mathsf{T}^2$. Now let $U := D_1 \subset X_p$ and $V := \mathsf{T}^2 \subset X_p$, so that $U \cup V = X_p$ and $U \cap V = \partial D_1$. Let $i : U \cap V \hookrightarrow U$ and $j : U \cap V \hookrightarrow V$ be the canonical inclusions.

  Firstly, the induced map $j_* : \pi_1(U \cap V) \to \pi_1(U)$ is trivial since U is a contractible disc. Next, observe that $\pi_1(U \cap V) \cong \pi_1(\mathsf{S}^1) \cong \mathbb{Z}$ is generated by a single loop u, and $\pi_1(V) \cong \pi_1(\mathsf{T}^2) \cong \mathbb{Z}^{\oplus 2}$ is generated by a meridianal loop x and a lateral loop y. Say w.l.o.g. that D_1 is the disc glued onto the corresponding meridianal circle of T^2. Then the induced map $i_* : \pi_1(U \cap V) \to \pi_1(V)$ sends u to x^p, so by van Kampen

$$\pi_1(X_p) \cong \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V) \cong \frac{1 * \langle x, y \rangle}{\langle i_*(u) j_*(u)^{-1} \rangle} \cong \frac{\langle x, y \rangle}{\langle x^p \rangle}.$$

- Now observe that $X_{pq} \cong (X_p \coprod D_2)/\sim$, where we identify each point $e^{i\phi} \in \partial D_2, 0 \leq \phi < 2\pi$, with $(1, e^{iq\phi}) \in \mathsf{T}^2$. Let $R := D_2 \subset X_{pq}$ and $S := X_p \subset X_{pq}$, so that $R \cup S = X_{pq}$ and $R \cap S = \partial D_2$. Then similarly to the above,

$$\pi_1(X_{pq}) \cong \pi_1(R) *_{\pi_1(R \cap S)} \pi_1(S) \cong \frac{1 * (\langle x, y \rangle / \langle x^p \rangle)}{\langle y^q \rangle} \cong \frac{\langle x, y \rangle}{\langle x^p, y^q \rangle} \cong \mathbb{Z}_p \oplus \mathbb{Z}_q.$$

$\square$

Problem 3.

The universal cover $\pi : \mathbb{R} \twoheadrightarrow \mathsf{S}^1$ satisfies $\pi_*(\pi_1(\mathbb{R})) \supset f_*(\pi_1(X))$ since both of $\pi_1(\mathbb{R}), \pi_1(X)$ are trivial, and thus we have the lifting diagram on the right

$$\begin{tikzcd}
 & \pi_1(\mathbb{R}) \cong 1 \arrow[d, "\pi_*"] \\
\pi_1(X) \cong 1 \arrow[r, "f_*"] & \pi_1(\mathsf{S}^1),
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathbb{R} \arrow[d, two heads, "\pi"] \\
X \arrow[ur, dashed, "\exists\, \tilde{f}"] \arrow[r, "f"] & \mathsf{S}^1.
\end{tikzcd}$$

Let $\{h_t\}_{0 \leq t \leq 1}$ be a homotopy with $h_0 = \mathsf{id}_{\mathbb{R}}$ and $h_1 = c$ for some constant map $c : \mathbb{R} \to \mathbb{R}$. Then $\{h_t \circ \tilde{f}\}_{0 \leq t \leq 1}$ gives a homotopy between $\tilde{f} : X \to \mathbb{R}$ and the constant map $c : X \to \mathbb{R}$. We likewise have a lift $\tilde{g} : X \to \mathbb{R}$, together with a homotopy between $\tilde{g}$ and c. So $\tilde{f}$ and $\tilde{g}$ are related by some homotopy $\{k_t\}_{0 \leq t \leq 1}$ with $k_0 = \tilde{f}$ and $k_1 = \tilde{g}$, and then $\{\pi \circ k_t\}_{0 \leq t \leq 1}$ is a homotopy between f and g. $\square$

Problem 4.

Let $X := \mathsf{S}^1 \times \mathsf{D}^2$ be the solid torus and $A := \mathsf{S}^1 \times \partial\mathsf{D}^2$ its boundary; then $X \cong \mathsf{S}^1$ and $A \cong \mathsf{T}^2$, so

$$\mathsf{H}_j(X) \cong \begin{cases} \mathbb{Z} & j = 0, 1, \\ 0 & \text{else,} \end{cases} \quad \mathsf{H}_j(A) \cong \begin{cases} \mathbb{Z} & j = 0, \\ \mathbb{Z}^{\oplus 2} & j = 1, \\ \mathbb{Z} & j = 2, \\ 0 & \text{else.} \end{cases}$$

By the long exact sequence $\cdots \to \mathsf{H}_j(A) \to \mathsf{H}_j(X) \to \mathsf{H}_j(X, A) \to \mathsf{H}_{j-1}(A) \to \cdots$ for relative homology, we have

$$0 \to \mathsf{H}_3(X, A) \to \mathbb{Z} \to 0 \to \mathsf{H}_2(X, A) \xrightarrow{\delta_2} \mathbb{Z}^{\oplus 2} \xrightarrow{\iota_1} \mathbb{Z} \xrightarrow{\kappa_1} \mathsf{H}_1(X, A) \xrightarrow{\delta_1} \mathbb{Z} \xrightarrow{\iota_0} \mathbb{Z} \xrightarrow{\kappa_0} \mathsf{H}_0(X, A) \to 0$$

and we calculate the relative homologies as follows.

- Immediately, $\mathsf{H}_3(X, A) \cong \mathbb{Z}$.
- $\mathsf{H}_1(A)$ is generated by a lateral loop $[x] \in \mathsf{H}_1(\mathsf{S}^1)$ and a meridianal loop $[y] \in \mathsf{H}_1(\partial\mathsf{D}^2)$. The inclusion $\iota : A \hookrightarrow X$ maps x to the same lateral loop, so that $\iota_1([x])$ is the single generator of $\mathsf{H}_1(X) \cong \mathbb{Z}$, but includes y into the contractible component D^2, whereby $\iota_1([x]) = 1$ and $\iota_1([y]) = 0$. Thus we have $\mathsf{im}(\delta_2) \cong \mathsf{ker}(\iota_1) \cong \mathbb{Z}$, and also $\mathsf{ker}(\delta_2) \cong 0$, so $\mathsf{H}_2(X, A) \cong \mathbb{Z}$.
- By the above, $\mathsf{ker}(\kappa_1) \cong \mathsf{im}(\iota_1) \cong \mathbb{Z}$, and so $\mathsf{ker}(\delta_1) \cong \mathsf{im}(\kappa_1) \cong 0$. Moreover ι_0 is injective since it's induced by the inclusion $\iota : A \hookrightarrow X$ of path connected spaces, so $\mathsf{im}(\delta_1) \cong \mathsf{ker}(\iota_0) \cong 0$. Thus $\mathsf{H}_1(X, A) \cong 0$.
- We now have $\mathsf{ker}(\kappa_0) \cong \mathsf{im}(\iota_0) \cong \mathbb{Z}$ since $\mathsf{ker}(\iota_0) \cong 0$. Then $\mathsf{im}(\kappa_0) \cong 0$, and since κ_0 is surjective, then $\mathsf{H}_0(X, A) \cong 0$.

Hence $\mathsf{H}_j(X, A) \cong \begin{cases} 0 & j = 0, 1, \\ \mathbb{Z} & j = 2, 3, \\ 0 & \text{else.} \end{cases}$ □

Problem 5.

For each $1 \leq j \leq n$, let θ_j be an angular coordinate for the j-th S^1 component of $\mathsf{T}^n \cong \prod^n \mathsf{S}^1$. Then $\mathsf{d}\theta_j$ is a closed 1-form on T^n, and $f^*\mathsf{d}\theta_j$ is a closed 1-form on M, with $[f^*\mathsf{d}\theta_j] = 0 \in \mathsf{H}^1_{\mathsf{dR}}(M) \cong 0$. So $[(f^*\mathsf{d}\theta_1) \wedge \ldots \wedge (f^*\mathsf{d}\theta_n)] = 0 \in \mathsf{H}^n_{\mathsf{dR}}(M)$, and

$$0 = \int_M (f^*\mathsf{d}\theta_1) \wedge \ldots \wedge (f^*\mathsf{d}\theta_n) = \int_M f^*(\mathsf{d}\theta_1 \wedge \ldots \wedge \mathsf{d}\theta_n) = \mathsf{deg}(f) \underbrace{\int_{\mathsf{T}^n} \mathsf{d}\theta_1 \wedge \ldots \wedge \mathsf{d}\theta_n}_{\neq 0},$$

where the integral on the right is nonzero since $\mathsf{d}\theta_1 \wedge \ldots \wedge \mathsf{d}\theta_n$ is a volume form on T^n. □

Problem 6.

Remark. We can actually do this more generally. Let $m, n \in \mathbb{N}$, denote by $\mathsf{Mat}_{m \times n}(\mathbb{R})$ the vector space of all $m \times n$ matrices, and denote by $X \subset \mathsf{Mat}_{m \times n}(\mathbb{R})$ the subset of those matrices having rank $k \in \mathbb{N}$.

Denote by $X' \subset \mathsf{Mat}_{m\times n}(\mathbb{R})$ the submanifold of those block matrices $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ whose upper-left $k \times k$ block a is invertible. Any matrix $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in X'$, written in the form above, has rank k if and only if the product

$$\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{m\times n}\underbrace{\begin{pmatrix} 1_{k\times k} & -a^{-1}b \\ 0 & 1_{(m-k)\times(m-k)} \end{pmatrix}}_{m\times m} = \underbrace{\begin{pmatrix} a & 0 \\ c & -ca^{-1}b+d \end{pmatrix}}_{n\times m}$$

has rank k, since the matrix we're multiplying by is invertible. Since a already has rank k, this requires the lower-right $(n-k)\times(m-k)$ block $-ca^{-1}b+d$ of the matrix on the right-hand side to be 0. Thus the space X'' of rank-k matrices belonging to X' can be identified with $f^{-1}(0)$, where f is the smooth map

$$f : X' \to \mathsf{Mat}_{(n-k)\times(m-k)}(\mathbb{R}), \quad f(x) := -ca^{-1}b + d,$$

with a, b, c, d corresponding to x as above. To conclude the proof, it's enough to show that X'' is a manifold, since matrices in X and matrices in X'' are related by (smooth) elementary row operations. Now, it's enough to check that 0 is a regular value of f. For any $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in f^{-1}(0)$, if $y \in \mathsf{Mat}_{(n-k)\times(m-k)}(\mathbb{R})$ is arbitrary, then defining

$$\alpha : [0,1] \to \mathsf{Mat}_{(n-k)\times(m-k)}(\mathbb{R}), \quad \alpha(t) := \begin{pmatrix} a & b \\ c & d \end{pmatrix} + t\begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix},$$

we see that

$$\mathsf{d}f_x \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix} = (f \circ \alpha)'(0) = (-ca^{-1}b + d + ty)'(0) = y,$$

whereby $\mathsf{d}f_x$ is surjective. Thus X'' is a submanifold of X', and in particular is a manifold. $\square$

Problem 7.

Suppose $\omega \in \Omega^1(\mathsf{S}^2)$ has $\phi^*\omega = \omega$ for every $\phi \in \mathsf{SO}(3)$. Then for arbitrary $x \in \mathsf{S}^2$ and $v \in \mathsf{T}_x\mathsf{S}^2$,

$$\omega_x(v) = \phi^*\omega_x(v) = \omega_{\phi(x)} \circ \mathsf{d}\phi_x(v)$$

for every $\phi \in \mathsf{SO}(3)$ by the definition of the pullback. Now $\mathsf{SO}(3)$ acts transitively on TS^2 by $\phi \cdot (y, w) := (\phi(y), \mathsf{d}\phi_y(w))$, so we may let $\phi \in \mathsf{SO}(3)$ above be such that $\phi \cdot (x, v) = (x, 0)$, and thus $\omega_x(v) = 0$. Hence $\omega \equiv 0$. $\square$

2007, Fall

Problem 1.

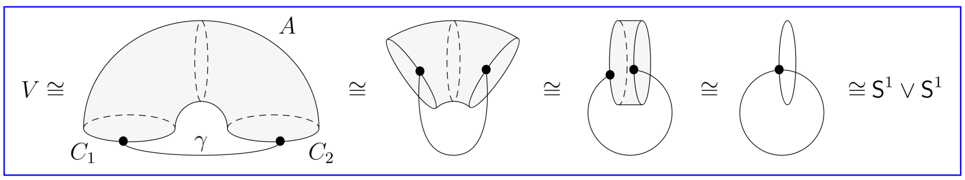
We already know that $\mathsf{H}_0(X \times \mathsf{S}^n) \cong 0$ by path connectedness. If $j \geq 1$, then by Künneth,

$$\mathsf{H}_j(X \times \mathsf{S}^n) \cong [\underbrace{\mathsf{H}_0(X)}_{\cong \mathbb{Z}} \otimes \mathsf{H}_j(\mathsf{S}^n)] \oplus [\mathsf{H}_j(X) \otimes \underbrace{\mathsf{H}_0(\mathsf{S}^n)}_{\cong \mathbb{Z}}] \oplus \bigoplus_{\substack{k,\ell \geq 1 \\ k+\ell = j}} \underbrace{\mathsf{H}_k(X)}_{\cong 0} \otimes \mathsf{H}_\ell(\mathsf{S}^n) \cong \mathsf{H}_j(X) \oplus \mathsf{H}_j(\mathsf{S}^n).$$

Thus $\mathsf{H}_j(X \times \mathsf{S}^n) \cong \begin{cases} \mathbb{Z} & j = 0, n, \\ 0 & \text{else.} \end{cases}$ □

Problem 2.

Let $U \cong \mathbb{R}^3$, and let V be the union of the attached handle and a curve γ connecting C_1 to C_2. Then U is equivalent to a wedge of two circles, and $U \cap V$ is the "pair of handcuffs" $C_1 \cup \gamma \cup C_2$.



Denote by $i : U \cap V \hookrightarrow U$ and $j : U \cap V \hookrightarrow V$ the canonical inclusions. Then $i_* : \pi_1(U \cap V) \to \pi_1(U)$ is trivial since $\pi_1(\mathbb{R}^3) \cong 1$, so it remains to determine $j_* : \pi_1(U \cap V) \to \pi_1(V)$. Thinking of C_1, C_2, and γ as oriented paths, we see that $\pi_1(U \cap V)$ is generated by the loops $[C_1]$ and $[\gamma * C_2 * \gamma^{-1}]$, and $\pi_1(V)$ is generated by $[C_1]$ and $[\gamma]$. The inclusion j sends $[C_1]$ to itself, but identifies $[C_1]$ and $[C_2]$, which are now connected by the 2-cell A. Hence $j_*([C_1]) = [C_1]$ and $j_*([\gamma * C_2 * \gamma^{-1}]) = [\gamma * C_1 * \gamma^{-1}]$. So by van Kampen

$$\pi_1(X) \cong \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V) \cong \frac{1 * (\mathbb{Z}\langle [C_1] \rangle * \mathbb{Z}\langle [\gamma] \rangle)}{\langle i_*([C_1]) j_*([C_1])^{-1}, i_*([\gamma * C_2 * \gamma^{-1}]) j_*([\gamma * C_1 * \gamma^{-1}])^{-1} \rangle}$$

$$\cong \frac{\langle [C_1], [\gamma] \rangle}{\langle [C_1]^{-1}, [\gamma * C_2 * \gamma^{-1}]^{-1} \rangle} \cong \left\langle x, y \mid x = yxy^{-1} = 1 \right\rangle \cong \left\langle y \mid yy^{-1} = 1 \right\rangle \cong \mathbb{Z}.$$

□

Problem 3.

We may think of det as a function $\mathbb{R}^{n^2} \to \mathbb{R}$, with $\mathbb{R}^{n^2}$ being coordinatized by $(x_{ij})_{1 \leq i,j \leq n}$. Then for the matrix $I = (x_{ij})_{1 \leq i,j \leq n} \in \mathsf{Mat}_n(\mathbb{R})$, denoting by $I_{ij} \in \mathsf{Mat}_{n-1}(\mathbb{R})$ the matrix obtained from I by deleting the i-th row and j-th column, for $1 \leq i, j \leq n$, we have

$$\mathsf{det}(I) = \sum_{1 \leq i,j \leq n} x_{ij} (-1)^{i+j} \mathsf{det}(I_{ij}) \implies \left(\frac{\partial}{\partial x_{ij}} \mathsf{det} \right)(I) = (-1)^{i+j} \mathsf{det}(I_{ij}) = \begin{cases} 0 & i \neq j, \\ 1 & i = j. \end{cases}$$

Hence for any matrix $v = (v)_{1 \leq i,j \leq n} \in \mathsf{T}_I \mathsf{Mat}_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$, we have

$$(\mathsf{d}(\mathsf{det}))_I(v) = \sum_{1 \leq i,j \leq n} \left(\frac{\partial}{\partial x_{ij}} \mathsf{det} \right)(I) v_{ij} = \sum_{j=1}^{n} v_{jj} = \mathsf{tr}(v),$$

where tr denotes the trace. Hence $(\mathsf{d}(\mathsf{det}))_I = \mathsf{tr}$ as maps from $\mathsf{T}_I \mathsf{Mat}_n(\mathbb{R}) \cong \mathsf{Mat}_n(\mathbb{R})$ to $\mathbb{R}$. □

Problem 4.

The sphere S^{n-1} is a deformation retract of $\mathbb{R}^n \setminus 0$ via the normalization map $u : \mathbb{R}^n \setminus 0 \twoheadrightarrow \mathsf{S}^{n-1}$ given by $u(x) := x/\|x\|$, so assuming that $0 \notin \mathsf{im}(F)$, we have a well defined composite $f^{-1} \circ u \circ F : M \to \partial M$ which fits into the diagram

$$\partial M \xhookrightarrow{\quad \iota \quad} M \xrightarrow{\quad F \quad} \mathbb{R}^n \setminus 0 \xrightarrow{\ u\ } \mathsf{S}^{n-1} \xrightarrow{\quad f^{-1} \quad} \partial M,$$

$$\mathsf{H}_{n-1}(\partial M) \xrightarrow{\ \iota_*\ } \mathsf{H}_{n-1}(M) \xrightarrow{\qquad\qquad (f^{-1}\circ u \circ F)_* \qquad\qquad} \mathsf{H}_{n-1}(\partial M).$$

Notice that the composite function $\partial M \to \partial M$ along the top row is $\mathsf{id}_{\partial M}$ since $F\big|_{\partial M} = f$, so the induced composite function $\mathsf{H}_{n-1}(\partial M) \to \mathsf{H}_{n-1}(\partial M)$ along the bottom row is certainly nonzero. However, the single generator $[\partial M] \in \mathsf{H}_{n-1}(\partial M)$ is clearly mapped to a boundary in $\mathsf{H}_{n-1}(M)$ by ι_*, and so

$$(f^{-1} \circ u \circ F)_* \circ \iota_*([\partial M]) = (f^{-1} \circ u \circ F)_*(0) = 0,$$

a contradiction. $\square$

Problem 5.

(a) We have that

$$\mathsf{d}\omega = \mathsf{d}\left(\frac{x}{4x^2+y^2}\right) \wedge \mathsf{d}y - \mathsf{d}\left(\frac{y}{4x^2+y^2}\right) \wedge \mathsf{d}x = \left[\frac{\partial}{\partial x}\left(\frac{x}{4x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{4x^2+y^2}\right)\right] \mathsf{d}x \wedge \mathsf{d}y$$
$$= \left[\frac{-4x^2+y^2}{(4x^2+y^2)^2} + \frac{4x^2-y^2}{(4x^2+y^2)^2}\right] \mathsf{d}x \wedge \mathsf{d}y = 0.$$

(Note that the denominators above are never 0 on Ω.) $\square$

(b) Consider the ellipse $X \subset \Omega$ defined by the equation $4x^2 + y^2 = 4^2$. If $\omega = \mathsf{d}\eta$ for some $\eta \in \Omega^0(\Omega)$, then by Stokes $\int_X \omega = \int_{\partial X} \eta = 0$ since $\partial X = \varnothing$. But parametrizing X by $x(t) := 2\mathsf{cos}(t)$ and $y(t) := 4\mathsf{sin}(t)$ for $0 \le t < 2\pi$, we have

$$\int_X \omega = \int_0^{2\pi} \frac{x(t)y'(t) - y(t)x'(t)}{4^2} \mathsf{d}t = \frac{1}{16}\int_0^{2\pi} 8\left[\mathsf{cos}^2(t) + \mathsf{sin}^2(t)\right] \mathsf{d}t = \pi,$$

so ω can't be exact on Ω. $\square$

Problem 6.

- The set

$$\varphi(C) = \{[x:y:1] \in \mathbb{R}\mathsf{P}^2 \mid y^2 - x^3 + x = 0\} = \left\{\left[\frac{x}{z} : \frac{y}{z} : 1\right] \,\middle|\, \frac{y^2}{z^2} - \frac{x^3}{z^3} + \frac{x}{z} = 0, z \in \mathbb{R} \setminus 0\right\}$$
$$= \{[x:y:z] \mid y^2 z - x^3 + xz^2 = 0, z \in \mathbb{R} \setminus 0\}.$$

  isn't closed since it doesn't include the case $z = 0$. Consider the equation $y^2z - x^3 + xz^2 = 0$ with $z = 0$; regardless of the choice of $y \in \mathbb{R}$, this equation yields $x = 0$, so the only element of $\mathbb{R}\mathsf{P}^2$ satisfying $y^2z - x^3 + xz^2 = 0$ which doesn't already belong to $\varphi(C)$ is $[0:1:0]$. Then defining $f : \mathbb{R}\mathsf{P}^2 \to \mathbb{R}$ by $f([x:y:z]) := y^2z - x^3 + xz^2$, we have that

$$f^{-1}(0) = \{[x:y:z] \mid y^2z - x^3 + xz^2 = 0\} = \varphi(C) \cup \{[0:1:0]\}.$$

This set is closed since it's the preimage of the closed point $\{0\} \subset \mathbb{R}$ under the continuous map f, and is also obviously the smallest closed set containing $\varphi(C)$. Thus $f^{-1}(0) = \overline{\varphi(C)}$.

- So to show that $\overline{\varphi(C)}$ is a submanifold of $\mathbb{R}\mathsf{P}^2$, we need only verify that 0 is a regular value of f. To see this, let $[x : y : z] \in f^{-1}(0)$ and consider the map $\mathsf{d}f_{[x:y:z]} : \mathsf{T}_{[x:y:z]}\mathbb{R}\mathsf{P}^2 \to \mathsf{T}_0\mathbb{R}$,

$$\mathsf{d}f_{[x:y:z]} = \begin{pmatrix} -3x^2 + z^2 & 2yz & y^2 + 2xz \end{pmatrix}.$$

  This linear map is surjective as long as one of its entries is nonzero. Now, at least one of x, y, z is nonzero by definition of $\mathbb{R}\mathsf{P}^2$, so we have the following cases.

  - Suppose $x \neq 0$. If $z = 0$ then $-3x^2 + z^2 \neq 0$. If $z \neq 0$ and $y = 0$ then $y^2 + 2xz \neq 0$. If $z \neq 0$ and $y \neq 0$ then $2yz \neq 0$.
  - Suppose $y \neq 0$. If $z \neq 0$ then $2yz \neq 0$. If $z = 0$ then $y^2 + 2xz \neq 0$.
  - Suppose $z \neq 0$. If $y \neq 0$ then $2yz \neq 0$. If $y = 0$ and $x \neq 0$ then $y^2 + 2xz \neq 0$. If $y = x = 0$ then $-3x^2 + z^2 \neq 0$.

  Hence 0 is indeed a regular value of f.

$\square$

Problem 7.

Let $y_0 := f(x_0) \in N$, and let $p : (\tilde{N}, \tilde{y}_0) \twoheadrightarrow (N, y_0)$ be the covering space corresponding to the subgroup $f_*(\pi_1(M, x_0)) \subset \pi_1(N, y_0)$. Then

$$k := [\pi_1(N, y_0) : p_*(\pi_1(\tilde{N}, \tilde{y}_0))] = [\pi_1(N, y_0) : f_*(\pi_1(M, x_0))],$$

and p is a k-sheeted covering of (M, x_0). So we're done if we can show that $k < \infty$. Assume that $k = \infty$. Now by definition of p, there exists a lift

$$\begin{tikzcd} & (\tilde{N}, \tilde{y}_0) \arrow[d, two heads, "p"] \\ (M, x_0) \arrow[r, "f"] \arrow[ur, dashed, "\exists\, \tilde{f}"] & (N, y_0). \end{tikzcd}$$

Since M is compact, then so is $\mathsf{im}(\tilde{f}) \subset \tilde{N}$. But $\tilde{N}$ is certainly noncompact since it's an ∞-sheeted covering space, and so $\tilde{f}$ is nonsurjective. Then $\mathsf{deg}(\tilde{f}) = 0$, and thus $\mathsf{deg}(f) = \mathsf{deg}(p)\mathsf{deg}(\tilde{f}) = 0$. But this is a contradiction since $\mathsf{H}_n(f) : \mathsf{H}_n(M) \to \mathsf{H}_n(N)$ is nontrivial by assumption. $\square$

2008, Spring

Incomplete: 3, 6(b).

Problem 1.

By assumption, G acts transitively on the fiber $p^{-1}(x_0)$, and hence the covering $p : \tilde{X} \twoheadrightarrow X$ is normal. Thus the subgroup $H := p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \subset \pi_1(X, x_0)$ is normal, and the quotient $\pi_1(X, x_0)/H$ is well defined. This yields the short exact sequence

$$1 \longrightarrow \pi_1(\tilde{X}, \tilde{x}_0) \xrightarrow{p_*} \pi_1(X, x_0) \longrightarrow \pi_1(X, x_0)/H \longrightarrow 1.$$

But also since the covering p is normal, then its group G of deck transformations is isomorphic to $\pi_1(X, x_0)/H$, and this completes the proof. $\square$

Problem 2.

We may write $\mathbb{R}^n \cong V \oplus V^\perp$, with the orthogonal projection $\pi : \mathbb{R}^n \twoheadrightarrow V$ being the identity on the component V and the zero map on the component $V^\perp$. Given $v \in \mathsf{T}_x\mathbb{R}^n \cong \mathbb{R}^n$, by definition of the tangent space $\mathsf{T}_x\mathbb{R}^n$, there's a curve $\gamma : (-1, 1) \to \mathbb{R}^n$ such that $\gamma(0) = x$ and $\gamma'(0) = v$. Decomposing $v = (v_1, v_2) \in V \oplus V^\perp$ and $\gamma = (\gamma_1, \gamma_2) : (-1, 1) \to V \oplus V^\perp$, we have

$$\mathsf{d}\pi_x(v) = (\pi \circ \gamma)'(0) = \pi(\gamma_1'(0), \gamma_2'(0)) = \pi(v_1, v_2) = v_1 = \pi(v),$$

and so $\mathsf{d}\pi_x = \pi : \mathbb{R}^n \to V$. In particular, this gives $\mathsf{ker}(\mathsf{d}\pi_x) = \mathsf{ker}(\pi) = V^\perp$. Now, $\pi\big|_M : M \to V$ is an immersion if and only if $(\mathsf{d}\pi\big|_M)_x : \mathsf{T}_xM \to \mathsf{T}_{\pi(x)}V$ is injective for every $x \in M$, i.e.

$$0 = \mathsf{ker}((\mathsf{d}\pi\big|_M)_x) = (\mathsf{T}_xM) \cap \mathsf{ker}(\mathsf{d}\pi_x) = (\mathsf{T}_xM) \cap V^\perp$$

for every $x \in M$. $\square$

Problem 4.

(a) We have $\mathsf{d}\alpha = 0$ since $\alpha \in \Omega^n(\mathsf{S}^n)$ is a volume form. Then $\mathsf{d}(f^*\alpha) = f^*\mathsf{d}\alpha = 0$, so the form $f^*\alpha \in \Omega^n(\mathsf{S}^{2n-1})$ is closed. But every closed n-form on S^{2n-1} is exact since $0 < n < 2n - 1$ implies $\mathsf{H}^n_{\mathsf{dR}}(\mathsf{S}^{2n-1}) \cong 0$. So $f^*\alpha = \mathsf{d}\beta$ for some $\beta \in \Omega^{n-1}(\mathsf{S}^{2n-1})$. $\square$

(b) Let $\beta' \in \Omega^{n-1}(\mathsf{S}^{2n-1})$ also satisfy $f^*\alpha = \mathsf{d}\beta'$. Then $\mathsf{d}(\beta' - \beta) = f^*\alpha - f^*\alpha = 0$, so $\beta' - \beta$ is closed. As above, every closed $(n-1)$-form on S^{2n-1} is exact, and so $\beta' - \beta = \mathsf{d}\gamma$ for some $\gamma \in \Omega^{n-2}(\mathsf{S}^{2n-1})$. Hence

$$\int_{\mathsf{S}^{2n-1}} \beta' \wedge \mathsf{d}\beta' = \int_{\mathsf{S}^{2n-1}} (\beta + \mathsf{d}\gamma) \wedge \mathsf{d}(\beta + \mathsf{d}\gamma) = \int_{\mathsf{S}^{2n-1}} \beta \wedge \mathsf{d}\beta + \int_{\mathsf{S}^{2n-1}} \mathsf{d}\gamma \wedge \mathsf{d}\beta.$$

We're done if we can show that the second integral on the right-hand side vanishes. And indeed,

$$\int_{\mathsf{S}^{2n-1}} \mathsf{d}\gamma \wedge \mathsf{d}\beta = \int_{\mathsf{S}^{2n-1}} \mathsf{d}(\gamma \wedge \mathsf{d}\beta) = \int_{\mathsf{B}^{2n}} \mathsf{d}^2(\gamma \wedge \mathsf{d}\beta) = 0$$

by Stokes. $\square$

Problem 5.

We have $\mathsf{d}\omega = 3\mathsf{d}x \wedge \mathsf{d}y \wedge \mathsf{d}z$, so

$$\int_{\mathsf{S}^2} \omega = \int_{\mathsf{B}^3} \mathsf{d}\omega = 3\int_{\mathsf{B}^3} \mathsf{d}x \wedge \mathsf{d}y \wedge \mathsf{d}z = 3\mathsf{vol}(\mathsf{B}^3) = 4\pi$$

by Stokes. $\square$

Problem 6 (?).

(a) Equip $\mathbb{R}\mathsf{P}^1$ with the usual pair of charts $\{(U,t),(V,s)\}$, for

$$U := \{[x:y] \in \mathbb{R}\mathsf{P}^1 \mid x \neq 0\}, \quad V := \{[x:y] \in \mathbb{R}\mathsf{P}^1 \mid y \neq 0\},$$

and $t : U \to \mathbb{R}, s : V \to \mathbb{R}$ the maps given by $t([x:y]) := y/x$ and $s([x:y]) := x/y$. Now suppose $\omega \in \Omega^1(\mathbb{R}\mathsf{P}^1)$ has $f^*\omega = P(x)\mathsf{d}x$. Then on the chart (U,t), we may write $\omega = F(t)\mathsf{d}t$ for some smooth function $F : \mathbb{R} \to \mathbb{R}$, and on this chart

$$P(x)\mathsf{d}x = f^*\omega = f^*(F(t)\mathsf{d}t) = F(t \circ f(x))\mathsf{d}(t \circ f(x)) = F(t([x:1]))\mathsf{d}(t([x:1])) = F\left(\frac{1}{x}\right)\mathsf{d}\left(\frac{1}{x}\right)$$
$$= -\frac{F(1/x)}{x^2}\mathsf{d}x \implies -\frac{F(1/x)}{x^2} = P(x).$$

Writing $P(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_0$ for some $a_0, \ldots, a_m \in \mathbb{R}$, this gives

$$F\left(\frac{1}{x}\right) = -x^2(a_m x^m + a_{m-1}x^{m-1} + \cdots + a_0) = -a_m x^{m+2} - a_{m-1}x^{m+1} - \cdots - a_0 x^2$$
$$\implies F(t) = -\frac{a_m}{t^{m+2}} - \frac{a_{m-1}}{t^{m+1}} - \cdots - \frac{a_0}{t^2}.$$

At the point $[1:0] \in U$, we have $t([0:1]) = 0/1 = 0$, and $F(0) = \infty$, contradicting F is smooth on the chart (U,t). $\square$

Problem 7.

Background. The manifold M, together with the *sheaf of rings* $\mathscr{C}^\infty_M$, is a *locally ringed space* $(M, \mathscr{C}^\infty_M)$. In this problem we prove that any maximal ideal $\mathscr{I}$ of the ring of *global sections* $\mathscr{C}^\infty(M)$ consists of functions vanishing at some point $x \in M$. The localization of $\mathscr{C}^\infty(M)$ at this point is isomorphic to the *stalk of* $\mathscr{C}^\infty_M$ *at* x, that is, $\mathscr{C}^\infty(M)_{\mathscr{I}} \cong \mathscr{C}^\infty_{M,x}$.

Write $M = \{x_\alpha\}_{\alpha \in A}$ and assume $\mathscr{I}$ isn't of the desired form. Then for every $\alpha \in A$, there's some $f_\alpha \in \mathscr{I}$ with $f_\alpha(x_\alpha) \neq 0$; by continuity, there's some open neighborhood $U_\alpha \subset M$ such that $f_\alpha\big|_{U_\alpha}$ is either strictly positive or strictly negative. By multiplying by the constant function $-1 \in \mathsf{C}^\infty(M)$ if necessary, we may assume w.l.o.g. that $f_\alpha\big|_{U_\alpha} > 0$. Since M is compact, we may choose a finite subcover $\{U_j\}_{j=1}^m$ of the open cover $\{U_\alpha\}_{\alpha \in A}$. Then $f := \sum_{j=1}^m f_j \in \mathscr{I}$ since $\mathscr{I}$ is an ideal, and $f > 0$ on all of M by design. So the function $1/f \in \mathsf{C}^\infty(M)$ is well defined and $f(1/f) = 1$, whereby $\mathscr{I} = (1) = \mathsf{C}^\infty(M)$. But this is impossible since $\mathscr{I} \subsetneq \mathsf{C}^\infty(M)$ by virtue of being a maximal ideal. $\square$

2008, Fall

Problem 1.

Background. Certain references may call $(\Omega^\bullet(M), \mathsf{d}_f^\bullet)$ the *f-twisted de Rham cochain complex of M*. In this problem we show that d_f is indeed a coboundary operator, and compute the 0th cohomology of this complex for $\mathbb{R}$.

(a) Observe that $\mathsf{d}f \wedge \mathsf{d}f = \mathsf{d}(f \wedge \mathsf{d}f) = -\mathsf{d}(\mathsf{d}f \wedge f) = -\mathsf{d}f \wedge \mathsf{d}f$, and so $\mathsf{d}f \wedge \mathsf{d}f = 0$. Then

$$\begin{aligned}\mathsf{d}_f^2\omega &= \mathsf{d}_f(\mathsf{d}\omega + \mathsf{d}f \wedge \omega) = \mathsf{d}(\mathsf{d}\omega + \mathsf{d}f \wedge \omega) + \mathsf{d}f \wedge (\mathsf{d}\omega + \mathsf{d}f \wedge \omega) \\ &= \underbrace{\mathsf{d}^2\omega}_{=0} + \underbrace{\mathsf{d}^2 f}_{=0} \wedge \omega - \mathsf{d}f \wedge \mathsf{d}\omega + \mathsf{d}f \wedge \mathsf{d}\omega + \underbrace{\mathsf{d}f \wedge \mathsf{d}f}_{=0} \wedge \omega = 0\end{aligned}$$

for any $\omega \in \Omega^j(M)$. □

(b) Suppose $g \in \ker((\mathsf{d}_f)_0) \subset \Omega^0(M) \cong \mathsf{C}^\infty(\mathbb{R})$. Then $0 = \mathsf{d}_f g = \mathsf{d}g + \mathsf{d}f \wedge \mathsf{d}g = \mathsf{d}g - g\mathsf{d}f$, so $g = \mathsf{d}g/\mathsf{d}f$ and hence $g = c_g e^f$ for some constant $c_g \in \mathbb{R}$. Conversely, any $g \in \mathsf{C}^\infty(\mathbb{R})$ of this form clearly satisfies $\mathsf{d}_f g = 0$. Hence the assignment $g \mapsto c_g$ is a one-to-one correspondence from $\mathsf{H}_f^0(\mathbb{R}) \cong \ker((\mathsf{d}_f)_0)$ to $\mathbb{R}$, which completes the argument. □

Problem 2.

We need only check that the map $f^* : \mathsf{H}_{\mathsf{dR}}^{m+n}(\mathsf{S}^m \times \mathsf{S}^n) \to \mathsf{H}_{\mathsf{dR}}^{m+n}(\mathsf{S}^{m+n})$ is trivial, since we know that $\mathsf{H}_{\mathsf{dR}}^j(\mathsf{S}^{m+n}) \cong 0$ for all $j \geq 1$ with $j \neq m+n$. Given volume forms $\alpha \in \Omega^m(\mathsf{S}^m)$ and $\beta \in \Omega^n(\mathsf{S}^n)$, the canonical projections $\pi_m : \mathsf{S}^m \times \mathsf{S}^n \twoheadrightarrow \mathsf{S}^m$ and $\pi_n : \mathsf{S}^m \times \mathsf{S}^n \twoheadrightarrow \mathsf{S}^n$ yield the two nonzero forms $\pi_m^*\alpha \in \Omega^m(\mathsf{S}^m \times \mathsf{S}^n)$ and $\pi_n^*\beta \in \Omega^n(\mathsf{S}^m \times \mathsf{S}^n)$. It's easy to verify that $(\pi_m^*\alpha) \wedge (\pi_n^*\beta)$ is a volume form on $\mathsf{S}^m \times \mathsf{S}^n$, whereby $[(\pi_m^*\alpha) \wedge (\pi_n^*\beta)]$ generates $\mathsf{H}_{\mathsf{dR}}^{m+n}(\mathsf{S}^m \times \mathsf{S}^n)$. Then f^* is trivial if it maps this generator to 0. To see this, recall that f^* is trivial on $\mathsf{H}_{\mathsf{dR}}^m(\mathsf{S}^m \times \mathsf{S}^n)$ and $\mathsf{H}_{\mathsf{dR}}^n(\mathsf{S}^m \times \mathsf{S}^n)$, whereby

$$f^*[(\pi_m^*\alpha) \wedge (\pi_n^*\beta)] = \underbrace{(f^*[\pi_m^*\alpha])}_{=0} \wedge \underbrace{(f^*[\pi_n^*\beta])}_{=0} = 0$$

as desired. □

Problem 3 (?).

Remark. It may be tempting to try to exhibit C as the preimage of 0 under $f(x, y) := y^2 - x^3$ and observe that $\mathsf{d}f_{(0,0)}$ is nonsurjective. However, this wouldn't prove that C isn't a submanifold of $\mathbb{R}^2$, but only that f was the wrong choice of function; *a priori* we may have that $C = g^{-1}(p)$ for some other smooth function g and some regular value p of g.

Assume that C is a submanifold of $\mathbb{R}^2$. Then by the implicit function theorem, on a sufficiently small neighborhood of the point $(0, 0) \in C$, we can write y as a function of x. By definition of C, this function must be $y = \pm x^{3/2}$. But on any neighborhood of 0 on the x-axis, this isn't a function since it assigns two values to any $x > 0$. □

Problem 4.

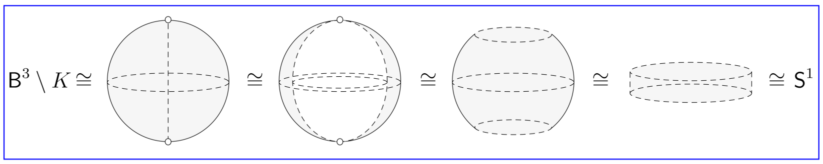
Let $X \subset \mathbb{R}^3$ be the solid torus with $\partial X = T$, and let $\omega := x\mathsf{d}y \wedge \mathsf{d}z - y\mathsf{d}x \wedge \mathsf{d}z + z\mathsf{d}x \wedge \mathsf{d}y$. Then $\mathsf{d}\omega = 3\mathsf{d}x \wedge \mathsf{d}y \wedge \mathsf{d}z$, and

$$\int_T \omega = 3\int_X \mathsf{d}x \wedge \mathsf{d}y \wedge \mathsf{d}z = 3\mathsf{vol}(X) = 3(2\pi R)(\pi r^2) = 6\pi^2 r^2 R$$

by Stokes. □

Problem 5.

We first stretch out the missing curve K inside B^3 until we hollow out the inside of the sphere, leaving us a copy of S^2 with two points removed. We then stretch out each of these missing points along the surface and toward the equator; the result is equivalent to a circle as shown.



Thus $\mathsf{H}_j(\mathsf{B}^3 \setminus K) \cong \mathsf{H}_j(\mathsf{S}^1) \cong \begin{cases} \mathbb{Z} & j = 0, 1, \\ 0 & \text{else.} \end{cases}$ $\square$

Problem 6.

The 3-sheeted covers of X are classified by equivalence classes of homomorphisms $\pi_1(X) \to \Sigma_3$. Note that if f is such a homomorphism, then f is completely determined by where it sends the two generators x, y of $\pi_1(X) \cong \pi_1(\mathsf{S}^1 \times \mathsf{S}^1) \cong \mathbb{Z}^{\oplus 2}$; and, $f(x)f(y) = f(xy) = f(yx) = f(y)f(x)$ since $\mathbb{Z}^{\oplus 2}$ is abelian. So these homomorphisms are in bijection with ordered pairs of elements $(\alpha, \beta) \in \Sigma_3$ with $\alpha\beta = \beta\alpha$, and we turn our attention to counting these pairs.

- Any of the six elements of Σ_3 commutes with itself, so this gives us six ordered pairs of the form (α, α), with $\alpha \in \Sigma_3$.
- Any of the six elements of Σ_3 commutes with $1 \in \Sigma_3$, so this gives us five new unordered pairs of the form $\{1, \alpha\}$, with $\alpha \in \Sigma_3$, and hence ten new ordered pairs. (We already counted the pair $(1, 1)$ in the previous step.)
- Finally, it is routinely verified that the two 3-cycles in Σ_3 commute, so we have two new ordered pairs $((123), (132))$ and $((132), (123))$.

In all, we've counted 18 pairs of the desired form, and from this we conclude that there are exactly 18 3-sheeted covers of X. $\square$

2009, Spring

Problem 1.

Take any $x = (x_1, x_2, x_3) \in \mathsf{S}^2$. Note that at least one of these coordinates is nonzero. We have

$$\mathsf{d}f_x = \begin{pmatrix} 2x_1 & -2x_2 & 0 \\ x_2 & x_1 & 0 \\ x_3 & 0 & x_1 \\ 0 & x_3 & x_2 \end{pmatrix}.$$

If $x_1 \neq 0$, then the last two columns are linearly independent. If $x_2 \neq 0$, then the first and last columns are linearly independent. And if $x_3 \neq 0$, then the first two columns are linearly independent. In any case, $\mathsf{rank}(\mathsf{d}f_x) \geq 2$. So since $2 = \mathsf{dim}_{\mathbb{R}}(\mathsf{T}_x\mathsf{S}^2) = \mathsf{rank}(\mathsf{d}f_x) + \mathsf{dim}_{\mathbb{R}}(\mathsf{ker}(\mathsf{d}f_x))$, we have that $\mathsf{ker}(\mathsf{d}f_x) = 0$, whereby f is an immersion.

Now, it's immediate that $f(x) = f(-x)$ for any $x \in \mathsf{S}^2$, whereby f descends to a well defined (surjective) immersion $\bar{f} : \mathbb{R}\mathsf{P}^2 \to f(\mathsf{S}^2)$, with $\mathbb{R}\mathsf{P}^2$ being the usual quotient $\mathsf{S}^2/\mathbb{Z}_2$. Recall that an embedding is a diffeomorphism onto its image, so we're done if we can show that $\bar{f}$ is an embedding; it remains only to verify that $\bar{f}$ is injective, and this can be (tediously) done directly from the definition of f. $\square$

Problem 2.

Since S^n is a deformation retract of $\mathbb{R}^{n+1} \setminus 0$ via the map $u : \mathbb{R}^{n+1} \twoheadrightarrow \mathsf{S}^n$ given by $u(x) := x/\|x\|$, we have an isomorphism $u^* : \mathsf{H}^n_{\mathsf{dR}}(\mathsf{S}^n) \to \mathsf{H}^n_{\mathsf{dR}}(\mathbb{R}^{n+1} \setminus 0)$. Moreover we have an isomorphism $I : \mathsf{H}^n_{\mathsf{dR}}(\mathsf{S}^n) \to \mathbb{R}$ given by $I([\omega]) := \int_{\mathsf{S}^n} \omega$, and so the composite

$$\mathsf{H}^n_{\mathsf{dR}}(\mathbb{R}^{n+1} \setminus 0) \xrightarrow{(u^*)^{-1}} \mathsf{H}^n_{\mathsf{dR}}(\mathsf{S}^n) \xrightarrow{\ I\ } \mathbb{R}$$

is an isomorphism. A closed n-form $\omega \in \Omega^n(\mathbb{R}^{n+1} \setminus 0)$ is exact if and only if $[\omega] = 0 \in \mathsf{H}^n_{\mathsf{dR}}(\mathbb{R}^{n+1} \setminus 0)$. By the above isomorphism, this is equivalent to $\int_{\mathsf{S}^n} \omega = I \circ (u^*)^{-1}([\omega]) = 0$. $\square$

Problem 3.

If $Z = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y}$ satisfies $[X, Z] = [Y, Z] = 0$, then

$$\begin{aligned}
0 = [X, Z] = XZ - ZX &= e^x \frac{\partial}{\partial x}\left(f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y}\right) + \left(f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y}\right) e^x \frac{\partial}{\partial x} \\
&= e^x \frac{\partial f}{\partial x}\frac{\partial}{\partial x} + e^x f \frac{\partial^2}{\partial x^2} + e^x \frac{\partial g}{\partial x}\frac{\partial}{\partial y} + e^x g \frac{\partial^2}{\partial x \partial y} - f e^x \frac{\partial^2}{\partial x^2} - f e^x \frac{\partial}{\partial x} - g e^x \frac{\partial^2}{\partial x \partial y} - 0 \\
&= e^x \left(\frac{\partial f}{\partial x} - f\right)\frac{\partial}{\partial x} + e^x \frac{\partial g}{\partial x}\frac{\partial}{\partial y}, \\
0 = [Y, Z] = YZ - ZY &= \frac{\partial}{\partial y}\left(f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y}\right) - \left(f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y}\right)\frac{\partial}{\partial y} \\
&= \frac{\partial f}{\partial y}\frac{\partial}{\partial x} + f\frac{\partial^2}{\partial x \partial y} + \frac{\partial g}{\partial y}\frac{\partial}{\partial y} + g\frac{\partial^2}{\partial y^2} - f\frac{\partial^2}{\partial x \partial y} - g\frac{\partial^2}{\partial y^2} \\
&= \frac{\partial f}{\partial y}\frac{\partial}{\partial x} + \frac{\partial g}{\partial y}\frac{\partial}{\partial y}.
\end{aligned}$$

The first equation gives $\frac{\partial f}{\partial x} = f$ and $\frac{\partial g}{\partial x} = 0$, and the second gives $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} = 0$. Therefore $f = c_1 e^x$ and $g = c_2$ for some constants $c_1, c_2 \in \mathbb{R}$, and $Z = c_1 e^x \frac{\partial}{\partial x} + c_2 \frac{\partial}{\partial y}$. $\square$

Problem 4.

For path connected spaces X, Y, we have $\pi_j(X \times Y) \cong \pi_j(X) \times \pi_j(Y)$ for all $j \in \mathbb{N}$. Hence

$$\pi_j(\mathsf{T}^p) \cong \prod^p \pi_j(\mathsf{S}^1) \cong \begin{cases} \mathbb{Z}^{\oplus p} & j = 1, \\ 0 & \text{else.} \end{cases}$$

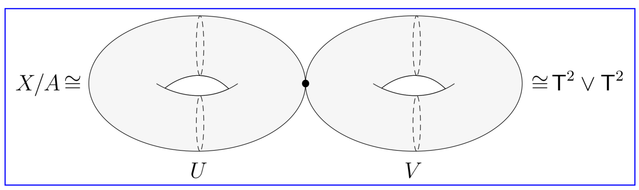
$\square$

Problem 5.

By problem 6 of 2006, Spring, we have $\mathbb{R}^3 \setminus K \cong \mathsf{T}^2$. Then $\pi_1(\mathbb{R}^3 \setminus K) \cong \pi_1(\mathsf{T}^2) \cong \mathbb{Z}^{\oplus 2}$. $\square$

Problem 6.

Note that (X, A) is a good pair since we can clearly find a neighborhood of A in X which deformation retracts to A, and so $\mathsf{H}_j(X, A) \cong \tilde{\mathsf{H}}_j(X/A)$ for each $j \in \mathbb{Z}$. By collapsing A to a point, we see that $X/A \cong \mathsf{T}^2 \vee \mathsf{T}^2$.



We decompose X/A into two tori U, V, with $U \cap V \cong *$, and by Mayer-Vietoris

$$0 \longrightarrow \mathbb{Z}^{\oplus 2} \longrightarrow \tilde{\mathsf{H}}_2(X/A) \longrightarrow 0 \longrightarrow \mathbb{Z}^{\oplus 4} \xrightarrow{k_1 - \ell_1} \tilde{\mathsf{H}}_1(X/A) \xrightarrow{\partial_1} \mathbb{Z} \xrightarrow{(i_0, j_0)} \mathbb{Z}^{\oplus 2} \xrightarrow{k_0 - \ell_0} \mathsf{H}_0(X/A) \longrightarrow 0$$

is exact. We compute the (reduced) homologies as follows.

- Immediately, $\tilde{\mathsf{H}}_2(X/A) \cong \mathbb{Z}^{\oplus 2}$.
- By exactness, $\ker(k_1 - \ell_1) \cong 0$. Now note that the map (i_1, j_1) is induced by the inclusions $i : U \cap V \hookrightarrow U$ and $j : U \cap V \hookrightarrow V$ of path connected spaces, and is hence injective. So $\mathsf{im}(\partial_1) \cong \ker(i_0, j_0) \cong 0$, whereby $\mathsf{im}(k_1 - \ell_1) \cong \ker(\partial_1) \cong \tilde{\mathsf{H}}_1(X/A)$. Thus $\tilde{\mathsf{H}}_1(X/A) \cong \mathbb{Z}^{\oplus 4}$.
- Next, $\ker(k_0 - \ell_0) \cong \mathsf{im}(i_0, j_0) \cong \mathbb{Z}$, so by exactness, $\mathsf{H}_0(X/A) \cong \mathsf{im}(k_0 - \ell_0) \cong \mathbb{Z}$.

Hence $\mathsf{H}_j(X, A) \cong \tilde{\mathsf{H}}_j(X/A) \cong \begin{cases} 0 & j = 0, \\ \mathbb{Z}^{\oplus 4} & j = 1, \\ \mathbb{Z}^{\oplus 2} & j = 2, \\ 0 & \text{else.} \end{cases}$ $\square$

2009, Fall

Problem 1.

(a) Let $p : \tilde{N} \twoheadrightarrow N$ be the cover corresponding to the subgroup $f_*(\pi_1(M)) \subset \pi_1(N)$. This cover has k sheets, where $k := \mathsf{deg}(p) = [\pi_1(N) : f_*(\pi_1(M))]$; note that k is a finite integer by assumption, and is nonzero since p is a covering map. By definition of p, there exists a lift

$$\begin{tikzcd} & \tilde{N} \arrow[d, two heads, "p"] \\ M \arrow[ur, dashed, "\exists\, \tilde{f}"] \arrow[r, "f"] & N, \end{tikzcd}$$

whereby $\mathsf{deg}(f) = \mathsf{deg}(p \circ \tilde{f}) = \mathsf{deg}(p)\mathsf{deg}(\tilde{f}) = [\pi_1(N) : f_*(\pi_1(M))]\mathsf{deg}(\tilde{f})$. □

(b) The antipodal map $a : \mathsf{S}^2 \to \mathsf{S}^2$ given by $a(x) := -x$ has $\mathsf{deg}(a) = -1$, but since S^2 is simply connected, we have $[\pi_1(\mathsf{S}^2) : a_*(\pi_1(\mathsf{S}^2))] = [1 : 1] = 1$. □

Problem 2.

No. Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ is differentiable and has $\mathsf{d}f(\frac{\partial}{\partial x}) = X, \mathsf{d}f(\frac{\partial}{\partial y}) = Y$. Then

$$x\frac{\partial}{\partial x} + \frac{\partial}{\partial y} = X = \mathsf{d}f\left(\frac{\partial}{\partial x}\right) = \frac{\partial f_1}{\partial x}\frac{\partial}{\partial x} + \frac{\partial f_2}{\partial x}\frac{\partial}{\partial y}, \quad -\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} = Y = \mathsf{d}f\left(\frac{\partial}{\partial y}\right) = \frac{\partial f_1}{\partial y}\frac{\partial}{\partial x} + \frac{\partial f_2}{\partial y}\frac{\partial}{\partial y}$$

and this in particular gives the system of equations

$$\frac{\partial f_2}{\partial x} = 1, \quad \frac{\partial f_2}{\partial y} = x.$$

The equation on the left gives $f_2(x, y) = x + g(y)$, for some function $g : \mathbb{R} \to \mathbb{R}$ of y, but the equation on the right gives $f_2(x, y) = xy + h(x)$, for some function $h : \mathbb{R} \to \mathbb{R}$ of x. These two expressions for f_2 can't agree on all of $\mathbb{R}^2$. □

Problem 3.

If there are no points $x \in \mathsf{S}^n$ with $f(x) = x$, then f is free of fixed points, and is thus homotopic to the antipodal map $a : \mathsf{S}^n \to \mathsf{S}^n$ given by $a(x) := -x$, by problem 3 of 2014, Fall. But then $\mathsf{deg}(f) = \mathsf{deg}(a) = (-1)^{n+1}$, a contradiction. Similarly if there are no points $x \in \mathsf{S}^n$ with $f(x) = -x$, then $-f$ is free of fixed points, and is therefore homotopic to a. Then

$$\mathsf{deg}(a)\mathsf{deg}(f) = \mathsf{deg}(a \circ f) = \mathsf{deg}(-f) = \mathsf{deg}(a) \implies \mathsf{deg}(f) = 1,$$

again a contradiction. □

Problem 4 (?)**.**

(a) Let $g : M \times \mathsf{B}^2 \to \mathbb{R}^n$ be given by $(x, y) \mapsto x - f(y)$, and observe that

$$\mathsf{im}(g) = \{v \in \mathbb{R}^n \mid \text{there are } x \in M, y \in \mathsf{B}^2 \text{ so } x = v + f(y)\} = \{v \in \mathbb{R}^n \mid T_v(\mathsf{im}(f)) \cap M \neq \varnothing\}.$$

Now for any $v \in \mathsf{im}(g)$ and any $(x,y) \in g^{-1}(v)$, the map $\mathsf{d}g_{(x,y)}(M \times \mathsf{B}^2) \to \mathsf{T}_v\mathbb{R}^n$ is nonsurjective since $\dim_{\mathbb{R}}(M) \leq n-3$ and $\dim_{\mathbb{R}}(\mathsf{B}^2) = 2$, so $\dim(\mathsf{im}(\mathsf{d}g_{(x,y)})) \leq n-1$. Hence v is a critical value of g. So by Sard, the complement

$$\mathbb{R}^n \setminus \mathsf{im}(g) = \{v \in \mathbb{R}^n \mid T_v(\mathsf{im}(f)) \cap M = \varnothing\}$$

contains all of the regular values of g, and thus has full measure in $\mathbb{R}^n$. Therefore $\mathbb{R}^n \setminus \mathsf{im}(g)$ contains arbitrarily small vectors. $\square$

(b) Take any $g : \mathsf{S}^1 \to \mathbb{R}^n \setminus M$; we're done if we can show that g is nullhomotopic. Consider a (continuous) map $f : \mathsf{B}^2 \to \mathbb{R}^n$ which glues $\partial\mathsf{B}^2$ onto $g(\mathsf{S}^1)$. Since M and $g(\mathsf{S}^1)$ are disjoint compact sets, then there exists an open neighborhood $U \supset g(\mathsf{S}^1)$ disjoint from M.

Thinking of B^2 as the closed unit disc in $\mathbb{C}$, then there's some $\epsilon \in (0, 1/2)$ small enough so that $f^{-1}(U)$ contains the "open collar"

$$C_{2\epsilon} := \{z \in \mathsf{B}^2 \mid \mathsf{dist}(z, \partial\mathsf{B}^2) < 2\epsilon\}.$$

Analogously defining the open collar $C_\epsilon \subset C_{2\epsilon}$, then $\mathsf{B}^2 \setminus C_\epsilon$ is itself homeomorphic to B^2. So by (a) there's an arbitrarily small vector $v \in \mathbb{R}^n$ such that $v + f(\mathsf{B}^2 \setminus C_\epsilon)$ is disjoint from M; since U is open and $f(C_{2\epsilon}) \subset U$, we may choose v small enough so that $v + f(C_{2\epsilon}) \subset U \subset \mathbb{R}^n \setminus M$.

Finally consider the homotopy $\{h_t : \mathsf{B}^2 \to \mathbb{R}^n \setminus M\}_{0 \leq t \leq 1}$ given by

$$h_t(z) := \begin{cases} t\epsilon^{-1}\mathsf{dist}(z, \partial\mathsf{B}^2)v + f(z) & z \in C_\epsilon, \\ tv + f(z) & z \in \mathsf{B}^2 \setminus C_\epsilon. \end{cases}$$

We see that $\{h_t\}_{0 \leq t \leq 1}$ is a homotopy between $h_0 = f$ and the map h_1 which pushes the "inner disc" $f(\mathsf{B}^2 \setminus C_\epsilon)$ by v, fixes the boundary $f(\mathsf{S}^1)$, and continuously connects these images by a collar which lies entirely in $U \subset \mathbb{R}^n \setminus M$.

But $h_1(\mathsf{B}^2)$ is the image of a (contractible) disc, mapped into $\mathbb{R}^n \setminus M$. Hence there exists a further homotopy $\{k_t : \mathsf{B}^2 \to \mathbb{R}^n \setminus M\}_{0 \leq t \leq 1}$ which contracts this image to a point $c \in h_1(\mathsf{B}^2)$, i.e. $k_0 = h_1$ and $k_1 = c$, where $c : \mathsf{B}^2 \to \mathbb{R}^n$ is the constant map to c. The composition of these two homotopies, restricted to the boundary $\partial\mathsf{B}^2$, is a nullhomotopy from g to c. $\square$

Problem 5.

Recall that S^{n-1} is a deformation retract of $\mathbb{R}^n \setminus 0$ via the normalization map $u : \mathbb{R}^{n+1} \setminus 0 \twoheadrightarrow \mathsf{S}^n$ given by $u(x) := x/\|x\|$. Hence we have an isomorphism $u^* : \Omega^n(S^n) \to \Omega^n(\mathbb{R}^{n+1} \setminus 0)$, so

$$\int_{\mathsf{S}^n} f^*\omega = \int_{\mathsf{S}^n} f^*u^*(u^*)^{-1}\omega = \int_{\mathsf{S}^n} (u \circ f)^*((u^*)^{-1}\omega) = \mathsf{deg}(u \circ f) \int_{\mathsf{S}^n} (u^*)^{-1}\omega$$

and similarly for g. Therefore as long as the denominator on the left is nonzero,

$$\frac{\int_{\mathsf{S}^n} f^*\omega}{\int_{\mathsf{S}^n} g^*\omega} = \frac{\mathsf{deg}(u \circ f) \int_{\mathsf{S}^n} (u^*)^{-1}\omega}{\mathsf{deg}(u \circ g) \int_{\mathsf{S}^n} (u^*)^{-1}\omega} = \frac{\mathsf{deg}(u \circ f)}{\mathsf{deg}(u \circ g)} \in \mathbb{Q}$$

since $\mathsf{deg}(u \circ f), \mathsf{deg}(u \circ g) \in \mathbb{Z}$. $\square$

Problem 6.

Observing that the solid genus-2 surface W is equivalent to $\bigvee^2 \mathsf{S}^1$,

$$\mathsf{H}_j(W) \cong \begin{cases} \mathbb{Z} & j=0, \\ \mathbb{Z}^{\oplus 4} & j=1, \\ \mathbb{Z} & j=2, \\ 0 & \text{else,} \end{cases} \quad \mathsf{H}_j(S) \cong \begin{cases} \mathbb{Z} & j=0, \\ \mathbb{Z}^{\oplus 2} & j=1, \\ 0 & \text{else.} \end{cases}$$

By the long exact sequence $\cdots \to \mathsf{H}_j(S) \to \mathsf{H}_j(W) \to \mathsf{H}_j(W,S) \to \mathsf{H}_{j-1}(S) \to \cdots$ for relative homology, we have

$$0 \to \mathsf{H}_3(W,S) \to \mathbb{Z} \to 0 \to \mathsf{H}_2(W,S) \xrightarrow{\delta_2} \mathbb{Z}^{\oplus 4} \xrightarrow{\iota_1} \mathbb{Z}^{\oplus 2} \xrightarrow{\kappa_1} \mathsf{H}_1(W,S) \xrightarrow{\delta_1} \mathbb{Z} \xrightarrow{\iota_0} \mathbb{Z} \xrightarrow{\kappa_0} \mathsf{H}_0(W,S) \to 0$$

and we calculate the relative homologies as follows.

- Immediately, $\mathsf{H}_3(W,S) \cong \mathbb{Z}$.
- $\mathsf{H}_1(S)$ is generated by two lateral loops $[x_1],[x_2]$ and two meridianal loops $[y_1],[y_2]$. The natural inclusion $\iota : S \hookrightarrow W$ maps x_1, x_2 to themselves, so that $\iota_1([x_1]), \iota_1([x_2])$ generate $\mathsf{H}_1(W) \cong \mathbb{Z}^{\oplus 2}$, but includes y_1, y_2 into contractible meridianal discs of W, whereby we have $\iota_1([y_1]) = \iota_1([y_2]) = 0$. Thus $\mathsf{im}(\delta_2) \cong \mathsf{ker}(\iota_1) \cong \mathbb{Z}^{\oplus 2}$, and also $\mathsf{ker}(\delta_2) \cong 0$, so it follows that $\mathsf{H}_2(W,S) \cong \mathbb{Z}^{\oplus 2}$.
- By the above, $\mathsf{ker}(\kappa_1) \cong \mathsf{im}(\iota_1) \cong \mathbb{Z}^{\oplus 2}$, and so $\mathsf{ker}(\delta_1) \cong \mathsf{im}(\kappa_1) \cong 0$. Also ι_0 is injective since it's induced by the inclusion $\iota : W \hookrightarrow S$ of path connected spaces, so $\mathsf{im}(\delta_1) \cong \mathsf{ker}(\iota_0) \cong 0$. Thus $\mathsf{H}_1(W,S) \cong 0$.
- We now have $\mathsf{ker}(\kappa_0) \cong \mathsf{im}(\iota_0) \cong \mathbb{Z}$ since $\mathsf{ker}(\iota_0) \cong 0$. Then $\mathsf{im}(\kappa_0) \cong 0$, and since κ_0 is surjective, then $\mathsf{H}_0(W,S) \cong 0$.

Hence $\mathsf{H}_j(W,S) \cong \begin{cases} 0 & j=0,1, \\ \mathbb{Z}^{\oplus 2} & j=2, \\ \mathbb{Z} & j=3, \\ 0 & \text{else.} \end{cases}$ $\square$

Problem 7 (?)**.**

Remark. I think this problem is way too hard for a qualifying exam. Maybe there's an easier approach which didn't occur to me.

Let $n := \mathsf{dim}_{\mathbb{R}}(N)$. We begin with the following construction at some fixed point $x \in M$. Let $V \subset N$ be an open subset containing M.

- Since $M \subset N$ is a codimension-1 submanifold, we can find a connected chart (W, ϕ) centered at x in the maximal atlas of N such that $\phi(W \cap M) = \phi(W) \cap (\mathbb{R}^{n-1} \times 0)$. W.l.o.g., W was chosen small enough so that $W \subset V$. We have an induced homeomorphism

$$\phi_* : \bigwedge^{n-1} \mathsf{T}^*(W \cap M) \to \bigwedge^{n-1} \mathsf{T}^*\phi(W \cap M).$$

Now, $\bigwedge^{n-1} \mathsf{T}^*\phi(W \cap M)$ is a (locally trivial) rank-1 vector bundle and ϕ is a homeomorphism, so we may assume w.l.o.g. that W was also chosen small enough so that we have a trivialization

$$f : \bigwedge^{n-1} \mathsf{T}^*\phi(W \cap M) \to \phi(W \cap M) \times \mathbb{R}.$$

- By our choice of ϕ, we have a natural inclusion $\iota : \phi\big|_{W\cap M}(W \cap M) \hookrightarrow \phi(W) \cap (\mathbb{R}^{n-1} \times 0)$ of codimension 1. And also since $\phi(W)$ is open, we can find a sufficiently small connected neighborhood $\tilde{S} \subset \phi(W \cap M) \times \mathbb{R}$ of $\phi(W \cap M) \times 0$ such that we have an inclusion

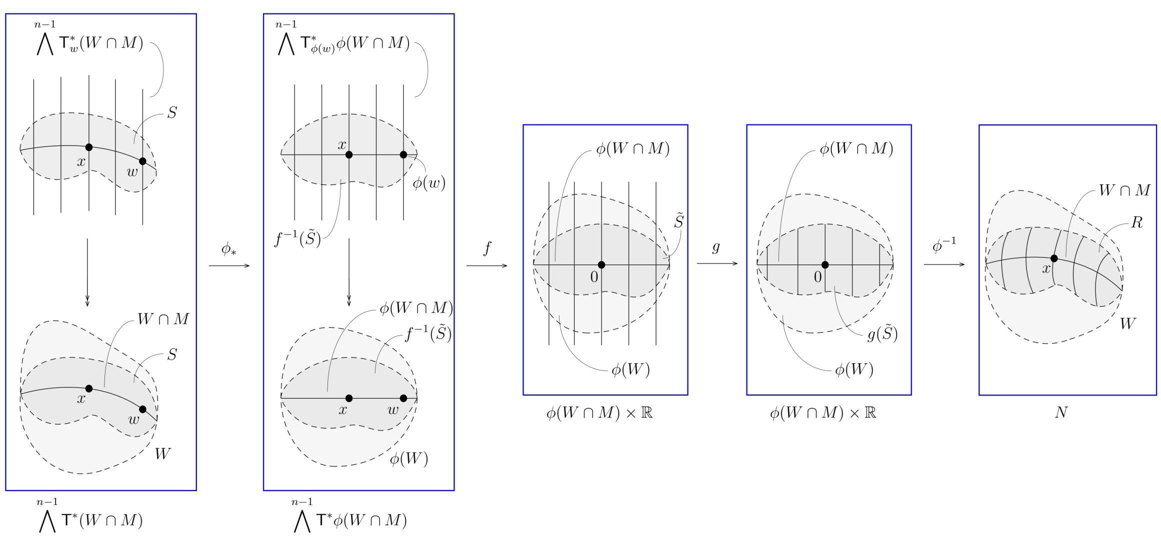
$$g := (\iota \times \mathsf{id}_{\mathbb{R}})\big|_{\tilde{S}} : \tilde{S} \hookrightarrow \phi(W),$$

  which further restricts to a homeomorphism $g : \tilde{S} \to g(\tilde{S})$, denoted again by g.

- Hence the composite $g \circ f \circ \phi_* : \bigwedge^{n-1} \mathsf{T}^*(W \cap M) \to \phi(W)$ restricts to homeomorphisms h and h_0 as in the commutative diagram below. Let S be the preimage $\phi_*^{-1} \circ f^{-1}(\tilde{S})$, let R be the preimage $\phi^{-1} \circ g(\tilde{S})$, and let $0_{W\cap M} : W \cap M \to \bigwedge^{n-1} \mathsf{T}^*(W \cap M)$ be the zero section.

$$\begin{tikzcd}
\bigwedge^{n-1} \mathsf{T}^*(W \cap M) \arrow[r, "g\circ f\circ\phi_*"] & \phi(W) \arrow[r, "\phi^{-1}"] & W \\
S \arrow[u, hook] \arrow[r, "h"] & g(\tilde{S}) \arrow[u, hook] \arrow[r, "\phi^{-1}|_{g(\tilde{S})}"] & R \arrow[u, hook] \\
S \setminus \mathsf{im}(0_{W\cap M}) \arrow[u, hook] \arrow[r, "h_0"] & g(\tilde{S}) \cap (\mathbb{R}^{n-1} \times 0) \arrow[u, hook] \arrow[r, "\phi^{-1}|_{g(\tilde{S})\times(\mathbb{R}^{n-1}\times 0)}"] & R \setminus M \arrow[u, hook]
\end{tikzcd}$$

We now repeat this construction at each point $x \in M$, writing x as a subscript for each of the maps and spaces above to keep track of the base points. Denote also by ϕ_x the restriction $\phi\big|_{R_x}$, for each $x \in M$.



We thus obtain a collection of charts $\{(R_x, \phi_x)\}_{x\in M}$ for M. Clearly $\{R_x\}_{x\in M}$ is an open cover for M, so we may choose a finite subcover $\{R_j\}_{j=1}^m$ by compactness of M, and consider the corresponding collection of charts $\{(R_j, \phi_j)\}_{j=1}^m$. Now $R_1, \ldots, R_m$ are open and connected, and M is also connected, so the union $U := \bigcup_{j=1}^m R_j \supset M$ is itself open and connected.

- Assume that M is orientable. Then $\mathsf{T}^*M \twoheadrightarrow M$ is orientable as a vector bundle, which means that the space $\left(\bigwedge^{n-1} \mathsf{T}^*M\right) \setminus \mathsf{im}(0_M)$ has exactly two connected components. Then

$$S_j \setminus \mathsf{im}(0_{W_j\cap M}) \subset \left(\bigwedge^{n-1} \mathsf{T}^*(W_j \cap M)\right) \setminus \mathsf{im}(0_{W_j\cap M})$$

  also has more than one connected component, for each $1 \le j \le m$. Hence via the composite homeomorphism $S_j \setminus \mathsf{im}(0_{W_j\cap M}) \to R_j \setminus M$ for each $1 \le j \le m$, the space

$$U \setminus M = \left(\bigcup_{j=1}^m R_j\right) \setminus M = \bigcup_{j=1}^m (R_j \setminus M)$$

  is also disconnected.

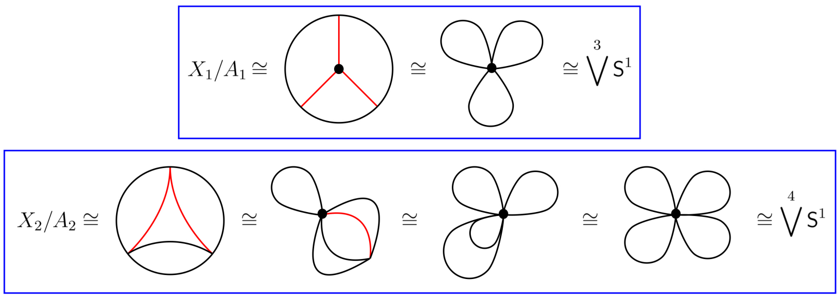
- Conversely, assume that for every open subset $V' \subset N$ containing M, there's a connected open subset $U' \subset V$ such that $U' \setminus M$ is disconnected. We take V to be the open subset U constructed above, and let $U' \subset U$ be a connected open subset such that $U' \setminus M$ is disconnected. Then upon patching together the disconnected images of the homeomorphisms $R_j \setminus M \to S_j \setminus \mathsf{im}(0_{W_j\cap M})$ for each $1 \le j \le m$, we see that $\left(\bigwedge^{n-1} \mathsf{T}^*M\right) \setminus \mathsf{im}(0_M)$ is disconnected. Choosing a connected component of this space specifies an orientation on M.

$\square$

2010, Fall

Problem 1.

If X is a CW complex (for instance, a graph) and $A \subset X$ a contractible subcomplex, then the natural quotient map $X \twoheadrightarrow X/A$ is a homotopy equivalence, whereby $\pi_1(X) \cong \pi_1(X/A)$. We satisfy these assumptions by letting $A_1 \subset X_1$ be the union of the three inner spokes, and $A_2 \subset X_2$ the union of two of the inner segments, as below.



Hence $\pi_1(X_1) \cong \pi_1\left(\bigvee^3 \mathsf{S}^1\right) \cong \mathsf{F}_3$ and $\pi_1(X_2) \cong \pi_1\left(\bigvee^4 \mathsf{S}^1\right) \cong \mathsf{F}_4$. $\square$

Problem 2.

By problem 3 of 2006, Spring, $X \cong \mathsf{S}^1 \vee \mathsf{S}^1 \vee \mathsf{S}^2$. Defining $U \cong \mathsf{S}^1 \vee \mathsf{S}^1, V \cong \mathsf{S}^2$ gives $U \cap V \cong *$, and

$$\mathsf{H}_j(U) \cong \begin{cases} \mathbb{Z} & j = 0, \\ \mathbb{Z}^{\oplus 2} & j = 1, \\ 0 & \text{else,} \end{cases} \quad \mathsf{H}_j(V) \cong \begin{cases} \mathbb{Z} & j = 0, 2, \\ 0 & \text{else.} \end{cases}$$

We already know that $\mathsf{H}_0(X) \cong \mathbb{Z}$ since X is path connected. Then by Mayer-Vietoris,

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathsf{H}_2(X) \longrightarrow 0 \longrightarrow \mathbb{Z}^{\oplus 2} \xrightarrow{k_1 - \ell_1} \mathsf{H}_1(X) \xrightarrow{\partial_1} \mathbb{Z} \xrightarrow{(i_0, j_0)} \mathbb{Z}^{\oplus 2} \longrightarrow \mathbb{Z}$$

is exact.

- Immediately, $\mathsf{H}_2(X) \cong \mathbb{Z}$.
- By exactness, $\ker(k_1 - \ell_1) \cong 0$, so $\ker(\partial_1) \cong \operatorname{im}(k_1 - \ell_1) \cong \mathbb{Z}^{\oplus 2}$. Next, note that (i_0, j_0) is injective since it's induced by the inclusions $i : U \cap V \hookrightarrow U$ and $j : U \cap V \hookrightarrow V$ of path connected spaces, so $\operatorname{im}(\partial_1) \cong \ker(i_0, j_0) \cong 0$. Thus $\mathsf{H}_1(X) \cong \mathbb{Z}^{\oplus 2}$.

Hence $\mathsf{H}_j(X) \cong \begin{cases} \mathbb{Z} & j = 0, \\ \mathbb{Z}^{\oplus 2} & j = 1, \\ \mathbb{Z} & j = 2, \\ 0 & \text{else.} \end{cases}$ $\square$

Problem 3 (?).

No. Suppose $\Sigma \subset \mathbb{R}^3$ is a compact immersed surface without boundary and satisfies $K(x) = -1$ for all $x \in \Sigma$. Then by Gauss-Bonnet,

$$-\mathsf{area}(\Sigma) = -\iint_\Sigma \mathsf{d}A = \iint_\Sigma K\mathsf{d}A = 2\pi\chi(\Sigma) = 2\pi(2-2g),$$

where g is the genus of Σ. Thus $-2\pi(2-2g) = \mathsf{area}(\Sigma) \geq 0$, and so we must have $g \geq 1$. But it's well known that any surface with genus $g \geq 1$ contains points having positive Gaussian curvature, so we've reached a contradiction. □

Problem 4.

Background. The *orthogonal group* $\mathsf{O}(n) \subset \mathsf{Mat}_n(\mathbb{R})$ is the group of isometries of $\mathbb{R}^n$, that is, the group of those matrices $x \in \mathsf{Mat}_n(\mathbb{R})$ which preserve the dot product, $\langle x\cdot, x\cdot\rangle = \langle\cdot,\cdot\rangle$. It's the real counterpart of the *unitary group* $\mathsf{U}(n) \subset \mathsf{Mat}_n(\mathbb{C})$. In this problem we show that $\mathsf{O}(n)$ is a *Lie group.*

Consider the map $f : \mathsf{Mat}_n(\mathbb{R}) \to \mathsf{Sym}(n)$, where $\mathsf{Sym}(n)$ is the space of symmetric $n \times n$ matrices, given by $f(x) := xx^\mathsf{T}$. Then $\mathsf{O}(n) = f^{-1}(1)$. Since f is clearly smooth, we're done if we can show that 1 is a regular value of f. To this end, let $a \in f^{-1}(1)$. Then for any $x \in \mathsf{T}_a\mathsf{Mat}_n(\mathbb{R})$,

$$\begin{aligned}\mathsf{d}f_a(x) &= \lim_{h\to 0}\frac{f(a+hx)-f(a)}{h} = \lim_{h\to 0}\frac{aa^\mathsf{T} + hxa^\mathsf{T} + ahx^\mathsf{T} + h^2xx^\mathsf{T} - 1}{h} \\ &= \lim_{h\to 0}\left(xa^\mathsf{T} + ax^\mathsf{T} + hxx^\mathsf{T}\right) = xa^\mathsf{T} + ax^\mathsf{T}.\end{aligned}$$

The right-hand side is indeed in $\mathsf{Sym}(n)$ since taking its transpose leaves it unchanged. And, $\mathsf{d}f_a$ differential is surjective since for any $y \in \mathsf{Sym}(n)$,

$$\mathsf{d}f_a\left(\frac{1}{2}ya\right) = \frac{1}{2}y\underbrace{aa^\mathsf{T}}_{=1} + \frac{1}{2}\underbrace{aa^\mathsf{T}}_{=1}\underbrace{y^\mathsf{T}}_{=y} = y.$$

This shows that $\mathsf{O}(n)$ is a manifold. To find its dimension, observe that any matrix in $\mathsf{Sym}(n)$ is completely determined by its n diagonal entries and $\frac{1}{2}(n^2-n)$ entries in the upper triangle. So it follows that we have $\mathsf{dim}_\mathbb{R}(\mathsf{Sym}(n)) = n + \frac{1}{2}(n^2-n) = \frac{1}{2}n(n+1)$, and

$$\mathsf{dim}_\mathbb{R}(\mathsf{O}(n)) = \mathsf{dim}_\mathbb{R}(\mathsf{Mat}_n(\mathbb{R})) - \mathsf{dim}_\mathbb{R}(\mathsf{Sym}(n)) = n^2 - \frac{1}{2}n(n+1) = \frac{1}{2}n(n-1).$$

□

Problem 5.

Note that $\omega = \alpha$ on S^{n-1} since the denominator of α is identically 1 here. Then by Stokes,

$$\int_{\mathsf{S}^{n-1}}\alpha = \int_{\mathsf{S}^{n-1}}\omega = \int_{\mathsf{B}^n}\mathsf{d}\omega = \int_{\mathsf{B}^n}\mathsf{d}x_1\wedge\ldots\wedge\mathsf{d}x_n = \mathsf{vol}(\mathsf{B}^n) \neq 0.$$

If $\alpha = \mathsf{d}\beta$ for some $\beta \in \Omega^{n-2}(\mathbb{R}^n \setminus 0)$, then we obtain the contradiction

$$\int_{\mathsf{S}^{n-1}}\alpha = \int_{\mathsf{S}^{n-1}}\mathsf{d}\beta = \int_{\partial\mathsf{S}^{n-1}}\beta = 0.$$

□

Problem 6.

Suppose $X \in \mathfrak{X}(\mathbb{R}^{2n})$ satisfies $\iota_X \omega = \mathsf{d}f$. Then upon equating the two expressions

$$\iota_X \omega = \sum_{j=1}^{n} \iota_X(\mathsf{d}x_j \wedge \mathsf{d}y_j) = \sum_{j=1}^{n} [(\iota_X \mathsf{d}x_j) \wedge \mathsf{d}y_j - \mathsf{d}x_j \wedge (\iota_X \mathsf{d}y_j)]$$

and

$$\mathsf{d}f = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial x_j} \mathsf{d}x_j + \frac{\partial f}{\partial y_j} \mathsf{d}y_j \right),$$

we have $\mathsf{d}x_j(X) = \iota_X \mathsf{d}x_j = \frac{\partial f}{\partial y_j}$ and $\mathsf{d}y_j(X) = \iota_X \mathsf{d}y_j = -\frac{\partial f}{\partial x_j}$ for each $1 \le j \le n$, whereby

$$X = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial y_j} \frac{\partial}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial}{\partial y_j} \right).$$

Note that $\mathsf{d}\omega = 0$. Then $\mathcal{L}_X \omega = \iota_X \underbrace{\mathsf{d}\omega}_{=0} + \mathsf{d} \underbrace{\iota_X \omega}_{=\mathsf{d}f} = \mathsf{d}(\mathsf{d}f) = 0$ by Cartan. □

Problem 7.

(a) If $\alpha \in \mathsf{C}_p(X;\mathbb{Z})$ has $\partial \alpha = 0$, then α defines a homology class $[\alpha] \in \mathsf{H}_p(X;\mathbb{Z})$. Since $\mathsf{H}_p(X;\mathbb{Z})$ is a finite $\mathbb{Z}$-module, there's some $k \in \mathbb{Z} \setminus 0$ with $k[\alpha] = 0 \in \mathsf{H}_p(X;\mathbb{Z})$, or equivalently, $k\alpha = \partial\beta$ for some $\beta \in \mathsf{C}_{p+1}(X;\mathbb{Z})$. □

(b) The element $u \in \mathsf{C}^{p+1}(X;\mathbb{Z})$ defines a cohomology class $[u] \in \mathsf{H}^{p+1}(X;\mathbb{Q}) \cong 0$ since $\mathsf{d}u = 0$. Then $[u] = 0 \in \mathsf{H}^{p+1}(X;\mathbb{Q})$ and hence $u = \mathsf{d}w$ for some $w \in \mathsf{C}^p(X;\mathbb{Q})$. With α, β, k as above, we define a map $\tilde{L}_u : \mathsf{C}_p(X;\mathbb{Z}) \to \mathbb{Q}$ by

$$\tilde{L}_u(\alpha) := \frac{1}{k} u(\beta) := \frac{1}{k} \mathsf{d}w(\beta) = \frac{1}{k} w(\partial\beta) = \frac{1}{k} w(k\alpha) = w(\alpha).$$

Indeed for any pair β, k satisfying $k\alpha = \beta$, the right-hand side is dependent only on α, so $\tilde{L}_u$ is well defined. Moreover, suppose $\alpha' \in \mathsf{C}_p(X;\mathbb{Z})$ has $[\alpha] = [\alpha'] \in \mathsf{H}_p(X;\mathbb{Z})$. Then $\alpha - \alpha' = \partial\gamma$ for some $\gamma \in \mathsf{C}_{p+1}(X;\mathbb{Z})$, and so

$$\tilde{L}_u(\alpha) - \tilde{L}_u(\alpha') = w(\alpha - \alpha') = w(\partial\gamma) = \mathsf{d}w(\gamma) \in \mathbb{Z} \implies \left[\tilde{L}_u(\alpha)\right] = \left[\tilde{L}_u(\alpha')\right] \in \mathbb{Q}/\mathbb{Z}.$$

Thus we have an induced well defined map $L_u : \mathsf{H}_p(X;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$ given by $L_u([\alpha]) := [\tilde{L}_u(\alpha)]$. And since $w = \tilde{L}_u$ is a homomorphism, then so is L_u. □

2011, Spring

Problem 1.

We have $\mathsf{d}\omega = \mathsf{d}x_1 \wedge \mathsf{d}x_2 \wedge \mathsf{d}x_3 \wedge \mathsf{d}x_4$ by a simple computation, and so

$$\int_{\mathsf{S}^3} \omega = \int_{\mathsf{B}^4} \mathsf{d}\omega = \int_{\mathsf{B}^4} \mathsf{d}x_1 \wedge \mathsf{d}x_2 \wedge \mathsf{d}x_3 \wedge \mathsf{d}x_4 = \mathsf{vol}(\mathsf{B}^4)$$

by Stokes. □

Problem 2.

- Consider the smooth map $f : \mathbb{R}^6 \to \mathbb{R}^3$ given by

$$f(x,y) := (\underbrace{x_1^2 + x_2^2 + x_3^2}_{=\|x\|^2}, \underbrace{y_1^2 + y_2^2 + y_3^2}_{=\|y\|^2}, \underbrace{x_1y_1 + x_2y_2 + x_3y_3}_{=\langle x,y\rangle}).$$

  Then $M = f^{-1}(1,1,0)$ by definition. For any $(x,y) \in f^{-1}(1,1,0)$, consider the differential

$$\mathsf{d}f_{(x,y)} = \begin{pmatrix} 2x_1 & 2x_2 & 2x_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2y_1 & 2y_2 & 2y_3 \\ y_1 & y_2 & y_3 & x_1 & x_2 & x_3 \end{pmatrix}$$

  Now there must be $1 \le i,j \le 3$ such that $x_i, y_j \neq 0$ (since $\|x\|, \|y\| = 1$) with $i \neq j$ (since $\langle x,y\rangle = 0$). Then the first row is nonzero in the i-th column, the second row in the $(3+j)$-th column, and the last row in the j-th and $(3+i)$-th columns. Thus $(1,1,0)$ is a regular value of f, whereby M is an embedded 3-dimensional submanifold of $\mathbb{R}^6$.

- Since f is continuous, the preimage M of the closed point $(1,1,0)$ is closed. So to see that M is compact, it remains to check that M is bounded. But this is immediate since for any $(x,y) \in M$, we have $\|(x,y)\|^2 = \|x\|^2 + \|y\|^2 = 2$.

- Let $u : \mathbb{R}^3 \to \mathbb{R}$ be given by $u(x) := \|x\|^2$, so that $\mathsf{S}^2 = u^{-1}(1)$. Then at each point $x \in \mathbb{R}^3$, upon canonically identifying $\mathsf{T}_x\mathbb{R}^3 \cong \mathbb{R}^3$, we get

$$\mathsf{T}_x\mathsf{S}^2 = \ker(\mathsf{d}u_x) = \ker\begin{pmatrix} 2x_1 & 2x_2 & 2x_3 \end{pmatrix} = \{y \in \mathbb{R}^3 \mid \langle x,y\rangle = 0\},$$

  and so

$$M = \{(x,y) \in \mathbb{R}^6 \mid \|x\| = 1, \|y\| = 1, \langle x,y\rangle = 0\} \cong \{(x,y) \mid x \in \mathsf{S}^2, y \in \mathsf{T}_x\mathsf{S}^2, \|y\| = 1\}.$$

  The right-hand side is precisely the definition of the unit tangent bundle of S^2.

□

Problem 3.

(a) Firstly, $\mathbb{RP}^1 \cong \mathsf{S}^1$ and so $\pi_1(\mathbb{RP}^1) \cong \pi_1(\mathsf{S}^1) \cong \mathbb{Z}$. If now $n \geq 2$, then $\mathbb{RP}^n$ is the quotient of S^n by the antipodal action of $\mathbb{Z}_2$ defined by $1 \cdot x := x$ and $-1 \cdot x := -x$ for all $x \in \mathsf{S}^n$. Then $\mathbb{Z}_2$ is the group of deck transformations of the (normal, simply connected) universal cover $\mathsf{S}^n \twoheadrightarrow \mathbb{RP}^n$, and so $\pi_1(\mathbb{RP}^n) \cong \mathbb{Z}_2$. □

(b) We may construct S^n by starting with two 0-cells e^0_-, e^0_+, then gluing on two "half-circle" 1-cells e^1_-, e^1_+, then gluing on two "half-sphere" 2-cells e^2_-, e^2_+, etc., until we've glued on two "half-sphere" n-cells e^n_-, e^n_+. In the quotient $\mathbb{R}\mathsf{P}^n = \mathsf{S}^n/\mathbb{Z}_2$, we identify e^j_- and e^j_+ for each $0 \le j \le n$. Thus $\mathbb{R}\mathsf{P}^n$ consists of exactly one j-cell e^j (with attaching map the 2-fold cover $p_{j-1} : \mathsf{S}^{j-1} \twoheadrightarrow \mathbb{R}\mathsf{P}^{j-1}$) for each $0 \le j \le n$. □

(c) By (b), the cellular chain complex $(\mathsf{C}^{\mathsf{CW}}_\bullet(\mathbb{R}\mathsf{P}^n), \partial_\bullet)$ of $\mathbb{R}\mathsf{P}^n$ is given by

$$0 \longrightarrow \mathbb{Z}\langle e^n\rangle \xrightarrow{\partial_n} \mathbb{Z}\big\langle e^{n-1}\big\rangle \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} \mathbb{Z}\big\langle e^1\big\rangle \xrightarrow{\partial_1} \mathbb{Z}\big\langle e^0\big\rangle \longrightarrow 0.$$

Now fix some $1 \le j \le n$ and recall that the boundary map $\partial_j : \mathsf{C}^{\mathsf{CW}}_j(\mathbb{R}\mathsf{P}^n) \to \mathsf{C}^{\mathsf{CW}}_{j-1}(\mathbb{R}\mathsf{P}^n)$ is given by $\partial_j(e^j) = \mathsf{deg}(q_{j-1} \circ p_{j-1})e^{j-1}$, where q_{j-1} is the natural quotient map in the diagram

$$\mathsf{S}^{j-1} \xrightarrow{p_{j-1}} \mathbb{R}\mathsf{P}^{j-1} \xrightarrow{q_{j-1}} \mathbb{R}\mathsf{P}^{j-1}/\mathbb{R}\mathsf{P}^{j-2} \cong \mathsf{S}^{j-1}.$$

The restriction maps $q_{j-1} \circ p_{j-1}\big|_{e^{j-1}_-}, q_{j-1} \circ p_{j-1}\big|_{e^{j-1}_+}$ are homeomorphisms from the two hemispheres $e^{j-1}_-, e^{j-1}_+ \subset \mathsf{S}^{j-1}$, respectively, onto the space $\mathbb{R}\mathsf{P}^{j-1} \setminus \mathbb{R}\mathsf{P}^{j-2}$. Furthermore, letting $a : \mathsf{S}^{j-1} \to \mathsf{S}^{j-1}$ be the degree-$(-1)^j$ antipodal map, we have that

$$q_{j-1} \circ p_{j-1}\big|_{e^{j-1}_-} = q_{j-1} \circ p_{j-1}\big|_{e^{j-1}_+} \circ a,$$

and so

$$\mathsf{deg}(q_{j-1} \circ p_{j-1}) = \mathsf{deg}\left(q_{j-1} \circ p_{j-1}\big|_{e^{j-1}_-}\right) + \mathsf{deg}\left(q_{j-1} \circ p_{j-1}\big|_{e^{j-1}_+}\right) = (-1)^j + 1 = \begin{cases} 0 & j \text{ odd}, \\ 2 & j \text{ even}. \end{cases}$$

Thus if n is odd or even, then the cellular chain complex of $\mathbb{R}\mathsf{P}^n$ is given by

$$0 \longrightarrow \mathbb{Z}\langle e^n\rangle \xrightarrow{0} \mathbb{Z}\big\langle e^{n-1}\big\rangle \xrightarrow{2} \cdots \xrightarrow{2} \mathbb{Z}\big\langle e^1\big\rangle \xrightarrow{0} \mathbb{Z}\big\langle e^0\big\rangle \longrightarrow 0$$

or

$$0 \longrightarrow \mathbb{Z}\langle e^n\rangle \xrightarrow{2} \mathbb{Z}\big\langle e^{n-1}\big\rangle \xrightarrow{0} \cdots \xrightarrow{2} \mathbb{Z}\big\langle e^1\big\rangle \xrightarrow{0} \mathbb{Z}\big\langle e^0\big\rangle \longrightarrow 0,$$

respectively. From the sequences above, for $0 < j < n$,

$$\mathsf{H}^{\mathsf{CW}}_j(\mathbb{R}\mathsf{P}^n) \cong \frac{\mathsf{ker}(\partial_j)}{\mathsf{im}(\partial_{j+1})} \cong \begin{cases} \mathsf{ker}(0)/\mathsf{im}(2) & j \text{ odd}, \\ \mathsf{ker}(2)/\mathsf{im}(0) & j \text{ even} \end{cases} \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & j \text{ odd}, \\ 0 & j \text{ even}. \end{cases}$$

By path connectedness, $\mathsf{H}^{\mathsf{CW}}_0(\mathbb{R}\mathsf{P}^n) \cong \mathbb{Z}$. And,

$$\mathsf{H}^{\mathsf{CW}}_n(\mathbb{R}\mathsf{P}^n) \cong \mathsf{ker}(\partial_n) \cong \begin{cases} \mathbb{Z} & n \text{ odd}, \\ 0 & n \text{ even}. \end{cases}$$

It's clear that $\mathsf{H}^{\mathsf{CW}}_j(\mathbb{R}\mathsf{P}^n) \cong 0$ for all $j > n$. □

(d) $\mathbb{R}\mathsf{P}^n$ **is orientable if and only if** $n \geq 1$ **is odd**. Recall that a compact connected oriented (topological) n-manifold X without boundary has $\mathsf{H}_n(X;\mathbb{Z}) \cong \mathbb{Z}$. So by (c), $\mathbb{R}\mathsf{P}^n$ is unorientable if n is even. If n is odd, then a choice of connected component of $\mathsf{H}_n(\mathbb{R}\mathsf{P}^n;\mathbb{Z})\setminus 0 \cong \mathbb{Z}\setminus 0$ specifies an orientation on $\mathbb{R}\mathsf{P}^n$. $\square$

Problem 4.

See problem 5 of 2013, Fall, replacing g by f, and replacing f by a constant map.

Problem 5.

Remark. The argument below actually works for G an arbitrary connected topological group.

Since G is connected, it suffices to show that $\pi_1(G,1)$ is abelian. We're done if we can find, for any pair of loops $f, g : [0,1] \to G$ based at 1, a homotopy between $f \cdot g$ and $g \cdot f$, where $\cdot$ is the product in G. Consider the families of maps $\{u_t : [0,1] \to G\}_{0 \leq t \leq 1}$ and $\{v_t : [0,1] \to G\}_{0\leq t \leq 1}$ given by

$$u_t(s) := \begin{cases} f\left(\frac{2s}{t+1}\right) & 0 \leq s \leq \frac{1+t}{2}, \\ 1 & \frac{1+t}{2} \leq s \leq 1, \end{cases} \quad v_t(s) := \begin{cases} 1 & 0 \leq s \leq \frac{1-t}{2}, \\ g\left(\frac{2s+t-1}{t+1}\right) & \frac{1-t}{2} \leq s \leq 1. \end{cases}$$

Then the family of maps $\{w_t : [0,1] \to G\}_{0\leq t\leq 1}$ given by $w_t := u_t \cdot v_t$ yields a homotopy between $f * g$ and $f \cdot g$,

$$w_0 = u_0 \cdot v_0 = (f * 1) \cdot (1 * g) = (f \cdot 1) * (1 \cdot g) = f * g, \quad w_1 = u_1 \cdot v_1 = f \cdot g.$$

We may similarly construct a homotopy between $f * g$ and $g \cdot f$, and it follows that there exists a homotopy between $f \cdot g$ and $g \cdot f$. $\square$

Problem 6.

Firstly, there must be some point $x \in M$ with $K(x) > 0$, since M is a compact oriented surface of genus $g \geq 1$. But also (recalling that $\partial M = \varnothing$) we have by Gauss-Bonnet that

$$\iint_M K \,\mathsf{d}A = 2\pi\chi(M) = 2\pi(2-2g) \leq 0$$

since $g \geq 1$, and so $K \leq 0$ on some nonempty subset of M. In particular, there's some point $y \in M$ with $K(y) \leq 0$. Therefore, since K is continuous on M, there must be some point $z \in M$ with $K(z) = 0$ by the intermediate value theorem. $\square$

2012, Spring

Problem 1.

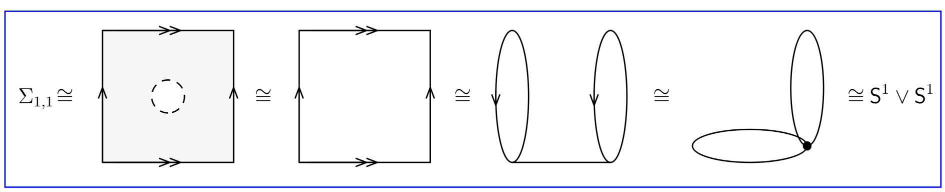
Let M be a compact n-manifold and suppose $f : M \to \mathbb{R}^n$ is an immersion; this in particular implies that $\mathsf{im}(f) \neq \varnothing$. We now have the following.

- Since M is closed and f is continuous, then $\mathsf{im}(f) \subset \mathbb{R}^n$ is closed.
- Since f is immersive, then for any $x \in M$, the map $\mathsf{d}f_x : \mathsf{T}_x M \to \mathsf{T}_{f(x)}\mathbb{R}^n$ is an injection between n-dimensional $\mathbb{R}$-vector spaces. Hence $\mathsf{d}f_x$ is an isomorphism, and so f is locally a diffeomorphism by the inverse function theorem. This implies that f is an open map, whereby $\mathsf{im}(f) \subset \mathbb{R}^n$ is open.

Thus $\mathsf{im}(f) \neq \varnothing$ is a simultaneously closed and open subspace of the connected space $\mathbb{R}^n$, whereby we must have $\mathsf{im}(f) = \mathbb{R}^n$. However this is impossible since the image of the compact space M under the continuous map f must be compact. $\square$

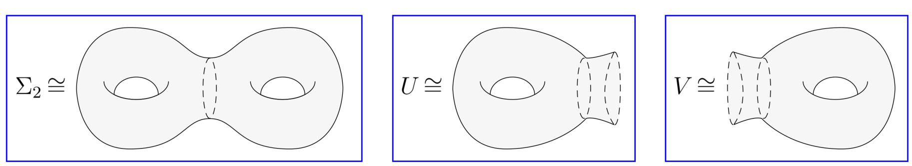
Problem 2.

(a) We stretch the missing disc inside the unit box outward until we're left with the box's (pre-identified) frame. Identifying the appropriate edges yields a wedge of two circles.



Hence $\mathsf{H}_j(\Sigma_{1,1}) \cong \mathsf{H}_j(\mathsf{S}^1 \vee \mathsf{S}^1) \cong \begin{cases} \mathbb{Z} & j = 0, \\ \mathbb{Z}^{\oplus 2} & j = 1, \\ 0 & \text{else.} \end{cases}$ $\square$

(b) Decomposing Σ_2 as the union of the punctured tori U and V below, we have that $U \cong V \cong \Sigma_{1,1}$ and $U \cap V \cong \mathsf{S}^1$.



Hence by (a),

$$\mathsf{H}_j(U) \cong \mathsf{H}_j(U) \cong \begin{cases} \mathbb{Z} & j = 0, \\ \mathbb{Z}^{\oplus 2} & j = 1, \\ 0 & \text{else,} \end{cases} \quad \mathsf{H}_j(U \cap V) \cong \begin{cases} \mathbb{Z} & j = 0, 1, \\ 0 & \text{else.} \end{cases}$$

We already know that $\mathsf{H}_0(\Sigma_2) \cong \mathbb{Z}$ since Σ_2 is path connected. Furthermore by Mayer-Vietoris,

$$0 \longrightarrow \mathsf{H}_2(\Sigma_2) \xrightarrow{\partial_2} \mathbb{Z} \xrightarrow{(i_1, j_1)} \mathbb{Z}^{\oplus 4} \xrightarrow{k_1 - \ell_1} \mathsf{H}_1(\Sigma_2) \xrightarrow{\partial_1} \mathbb{Z} \xrightarrow{(i_0, j_0)} \mathbb{Z}^{\oplus 2}$$

is exact.

- By exactness, $\mathsf{ker}(\partial_2) \cong 0$. Moreover, (i_1, j_1) is induced by the inclusions $i : U \cap V \hookrightarrow U$ and $j : U \cap V \hookrightarrow V$ of the boundary $U \cap V$ into either U or V, so $\mathsf{im}(i_1, j_1) \cong 0$. Hence we have $\mathsf{im}(\partial_2) \cong \mathsf{ker}(i_1, j_1) \cong \mathbb{Z}$, whereby $\mathsf{H}_2(\Sigma_2) \cong \mathbb{Z}$.
- By the above, $\mathsf{ker}(k_1 - \ell_1) \cong \mathsf{im}(i_1, j_2) \cong 0$, so $\mathsf{ker}(\partial_1) \cong \mathsf{im}(k_1 - \ell_1) \cong \mathbb{Z}^{\oplus 4}$. Note also that (i_0, j_0) is injective since it's induced by the inclusions i, j of path connected spaces, and so we have $\mathsf{im}(\partial_1) \cong \mathsf{ker}(i_0, j_0) \cong 0$. Thus $\mathsf{H}_1(\Sigma_2) \cong \mathbb{Z}^{\oplus 4}$.

Therefore $\mathsf{H}_j(\Sigma_2) \cong \begin{cases} \mathbb{Z} & j = 0, \\ \mathbb{Z}^{\oplus 4} & j = 1, \\ \mathbb{Z} & j = 2, \\ 0 & \text{else.} \end{cases}$ $\square$

Problem 3.

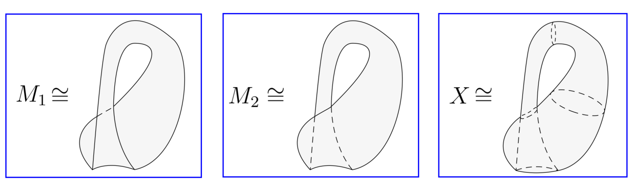
At any point $p \in S$, write $\omega_p = a_1 \mathsf{d}y \wedge \mathsf{d}z + a_2 \mathsf{d}x \wedge \mathsf{d}z + a_3 \mathsf{d}x \wedge \mathsf{d}y$ for constants $a_1, a_2, a_3 \in \mathbb{R}$, and write $e_j = (e_j^x, e_j^y, e_j^z)$ for each $j = 1, 2$. Then, using $n = (n_1, n_2, n_3) = e_1 \times e_2$, we have

$$\begin{aligned} n_1^2 + n_2^2 + n_3^2 = \|n\|^2 = 1 = \omega_p(e_1, e_2) &= a_1 \mathsf{d}y \wedge \mathsf{d}z(e_1, e_2) + a_2 \mathsf{d}x \wedge \mathsf{d}z(e_1, e_2) + a_3 \mathsf{d}x \wedge \mathsf{d}y(e_1, e_2) \\ = a_1(e_1^y e_2^z - e_1^z e_2^y) + a_2(e_1^x e_2^z - e_1^z e_2^x) &+ a_3(e_1^x e_2^y - e_1^y e_2^x) = a_1 n_1 + a_2(-n_2) + a_3 n_3. \end{aligned}$$

Comparing the left- and right-hand sides gives $a_1 = n_1, a_2 = -n_2, a_3 = n_3$. $\square$

Problem 4.

(a) Observe that X is the Klein bottle obtained by gluing M_1 and M_2 along their boundaries.



Firstly $M_1 \cong M_2 \cong \mathsf{S}^1$, so let x_1, x_2 be generators of $\pi_1(M_1) \cong \mathbb{Z}$ and $\pi_1(M_2) \cong \mathbb{Z}$, respectively. Moreover $M_1 \cap M_2 \cong \partial M_1 \cong \mathsf{S}^1$, so for each $j = 1, 2$, the inclusion $\iota_j : M_1 \cap M_2 = \partial M_j \hookrightarrow M_j$ winds the loop $1 \in \pi_1(M_1 \cap M_2) \cong \pi_1(\mathsf{S}^1) \cong \mathbb{Z}$ once over the "front" of M_j and once over the "back," so that $\iota_j(1) = x_j^2$. Then by van Kampen,

$$\pi_1(X) \cong \pi_1(M_1) *_{\pi_1(M_1 \cap M_2)} \pi_1(M_2) \cong \frac{\langle x_1, x_2 \rangle}{\langle \iota_1(1) \iota_2(1)^{-1} \rangle} \cong \langle x_1, x_2 \mid x_1^2 = x_2^2 = 1 \rangle.$$

$\square$

(b) **No**. The Klein bottle is unorientable. $\square$

Problem 5.

Equivalence classes of connected covers of $\mathbb{RP}^{14} \vee \mathbb{RP}^{15}$ are in bijection with the subgroups of

$$\pi_1(\mathbb{RP}^{14} \vee \mathbb{RP}^{15}) \cong \pi_1(\mathbb{RP}^{14}) * \pi_1(\mathbb{RP}^{15}) \cong \mathbb{Z}_2 * \mathbb{Z}_2 \cong \langle x, y \mid x^2 = y^2 = 1 \rangle.$$

The identity subgroup corresponds to the universal cover $\mathsf{S}^{14} \vee \mathsf{S}^{15}$; the entire group corresponds to the trivial cover $\mathbb{RP}^{14} \vee \mathbb{RP}^{15}$; the subgroups generated by x and y correspond to the covers $\mathbb{RP}^{14} \vee \mathsf{S}^{15}$ and $\mathsf{S}^{14} \vee \mathbb{RP}^{15}$, respectively. □

Problem 6.

By Cartan,

$$f^*(\mathcal{L}_Y \omega) = f^*(\iota_Y \mathsf{d}\omega) + f^*(\mathsf{d}\iota_Y \omega), \quad \mathcal{L}_X(f^*\omega) = \iota_X(\mathsf{d}f^*\omega) + \mathsf{d}\iota_X(f^*\omega).$$

We show that the right-hand sides are equal by showing equality of the corresponding summands. We have for the second summand

$$f^*(\mathsf{d}\iota_Y\omega) = \mathsf{d}f^*(\iota_Y\omega) = \mathsf{d}f^*(\omega(Y)) = \mathsf{d}(\omega(Y)\circ f) = \mathsf{d}(\omega(f_*(X))\circ f) = \mathsf{d}((f^*\omega)(X)) = \mathsf{d}\iota_X(f^*\omega).$$

Next, for any $x \in M$ and $v \in \mathsf{T}_x M$, we have for the first summand

$$\begin{aligned}(f^*(\iota_Y \mathsf{d}\omega))_x(v) &= (\iota_Y \mathsf{d}\omega)_{f(x)}(f_*v) = \mathsf{d}\omega_{f(x)}(Y(f(x)), f_*v) = \mathsf{d}\omega_{f(x)}(f_*(X(x)), f_*v) \\ &= (f^*\mathsf{d}\omega)_x(X(x), v) = (\iota_X(f^*\mathsf{d}\omega))_x(v) = (\iota_X(\mathsf{d}f^*\omega))_x(v),\end{aligned}$$

and so $f^*(\iota_Y \mathsf{d}\omega) = \iota_X(\mathsf{d}f^*\omega)$. □

Problem 7.

Remark. It's indeed the case that $[X, Y] = 0$, but this only tells us by Frobenius that the rank-2 distribution defined by X and Y (not the one orthogonal to X and Y) is integrable.

No. Denote by $\mathscr{D}$ the rank-2 distribution orthogonal to X and Y, and begin by taking some arbitrary vector field $V = v_1\frac{\partial}{\partial x_1} + v_2\frac{\partial}{\partial x_2} + v_3\frac{\partial}{\partial x_3} + v_4\frac{\partial}{\partial x_4} \in \mathscr{D}$. Then

$$\begin{aligned}\langle V, X\rangle = 0 &\implies v_1x_1 + v_2x_2 + v_3x_3 + v_4x_4 = 0, \\ \langle V, Y\rangle = 0 &\implies -v_1x_2 + v_2x_1 - v_3x_4 + v_4x_3 = 0,\end{aligned}$$

so we have the matrix equation

$$\begin{pmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} x_3 & x_4 \\ -x_4 & x_3 \end{pmatrix}\begin{pmatrix} v_3 \\ v_4 \end{pmatrix} = 0.$$

By assumption, $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \setminus 0$, so w.l.o.g. $x_1 \neq 0$. Then this equation gives

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = -\begin{pmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{pmatrix}^{-1}\begin{pmatrix} x_3 & x_4 \\ -x_4 & x_3 \end{pmatrix}\begin{pmatrix} v_3 \\ v_4 \end{pmatrix} = -\frac{1}{x_1^2 + x_2^2}\begin{pmatrix} x_1x_3 + x_2x_4 & x_1x_4 - x_2x_3 \\ x_2x_3 - x_1x_4 & x_1x_3 + x_2x_4 \end{pmatrix}\begin{pmatrix} v_3 \\ v_4 \end{pmatrix}.$$

Now, we may freely choose $(v_3, v_4) \in \mathbb{R}^2$ and this equation determines $(v_1, v_2) \in \mathbb{R}^2$ such that the resulting vector field V belongs to $\mathscr{D}$. Setting $(v_3, v_4) = -(x_1^2 + x_2^2, 0)$ and $(v_3', v_4') = -(0, x_1^2 + x_2^2)$, respectively, we obtain two sets of coefficients

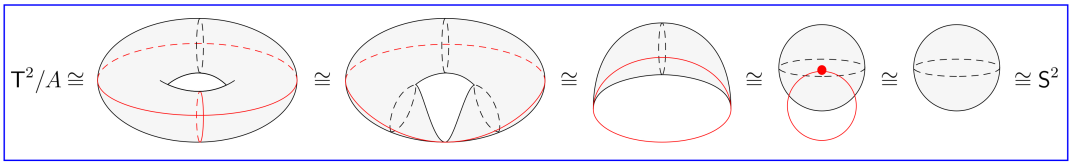
$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} x_1x_3 + x_2x_4 \\ x_2x_3 - x_1x_4 \\ -x_1^2 - x_2^2 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} v_1' \\ v_2' \\ v_3' \\ v_4' \end{pmatrix} = \begin{pmatrix} x_1x_4 - x_2x_3 \\ x_1x_3 + x_2x_4 \\ 0 \\ -x_1^2 - x_2^2 \end{pmatrix}$$

which yield, respectively, two vector fields $V, V' \in \mathscr{D}$. But now, $[V, V'] = -2(x_1^2 + x_2^2)Y$, whereby $[V, V'] \notin \mathscr{D}$ since $\langle -2(x_1^2 + x_2^2)Y, Y\rangle \neq 0$ for $x_1 \neq 0$. Thus $\mathscr{D}$ is nonintegrable by Frobenius. □

2012, Fall

Problem 1 (?)**.**

Observe that A is the union of a lateral circle and a meridianal one, and that the quotient T^2/A is equivalent to S^2 as shown below.



Hence $\mathsf{H}_j(\mathsf{T}^2/A) \cong \mathsf{H}_j(\mathsf{S}^2)$. Furthermore, A is a deformation retract of a small thickening of itself within T^2. So (T^2, A) is a good pair, whereby $\mathsf{H}^j(\mathsf{T}^2, A) \cong \tilde{\mathsf{H}}^j(\mathsf{T}^2/A)$ for each $j \geq 0$.

- By properties of reduced cohomology, $\mathsf{H}^0(\mathsf{T}^2, A) \cong \tilde{\mathsf{H}}^0(\mathsf{T}^2/A) \cong \mathsf{Hom}_{\mathbb{Z}}(\tilde{\mathsf{H}}_0(\mathsf{T}^2/A), \mathbb{Z}) \cong 0$, since $\tilde{\mathsf{H}}_0(\mathsf{T}^2/A) \cong 0$ by path connectedness of T^2/A.
- By Poincaré duality, $\mathsf{H}^1(\mathsf{T}^2, A) \cong \tilde{\mathsf{H}}^1(\mathsf{T}^2/A) \cong \mathsf{H}_1(\mathsf{T}^2/A) \cong \mathsf{H}_1(\mathsf{S}^2) \cong 0$.
- Similarly, $\mathsf{H}^2(\mathsf{T}^2, A) \cong \tilde{\mathsf{H}}^2(\mathsf{T}^2/A) \cong \mathsf{H}_0(\mathsf{T}^2/A) \cong \mathsf{H}_0(\mathsf{S}^2) \cong \mathbb{Z}$.
- And finally, for any $j \geq 3$, we have $\mathsf{H}^j(\mathsf{T}^2, A) \cong \tilde{\mathsf{H}}^j(\mathsf{T}^2/A) \cong \tilde{\mathsf{H}}^j(\mathsf{S}^2) \cong 0$.

In summary, $\mathsf{H}^j(\mathsf{T}^2, A) \cong \begin{cases} \mathbb{Z} & j = 2, \\ 0 & \text{else.} \end{cases}$ □

Problem 2.

Remark. This problem's description contains a mistake; the smash product of two (pointed) spaces X, Y is defined by $X \wedge Y := (X \times Y)/(X \vee Y)$. I'm not sure which "definition" of $\wedge$ this problem uses, so this I'll skip this one.

Problem 3.

(a) Recall that the cellular homology of X agrees with its usual singular homology. Let $(\mathsf{C}^{\mathsf{CW}}_\bullet(X), \partial_\bullet)$ denote the cellular chain complex of X,

$$0 \longrightarrow \mathsf{C}^{\mathsf{CW}}_2(X) \xrightarrow{\partial_2} \mathsf{C}^{\mathsf{CW}}_1(X) \xrightarrow{\partial_1} \mathsf{C}^{\mathsf{CW}}_0(X) \xrightarrow{\partial_0} 0$$

and $\mathsf{H}^{\mathsf{CW}}_\bullet(X)$ the homology of this complex. Name the 2-cells A, B, C, in the order that they're pictured.

- By path connectedness, we have $\mathsf{H}^{\mathsf{CW}}_0(X) \cong \mathbb{Z}$.
- We have that $\partial_1(a) = v - v = 0$ and $\partial_1(b) = b - b = 0$, so $\mathsf{ker}(\partial_1) = \mathbb{Z}\langle a, b\rangle$. Next,

$$\partial_2(A) = a - a = 0, \quad \partial_2(B) = 3b, \quad \partial_2(C) = a + b + a + b = 2(a + b),$$

so $\mathsf{im}(\partial_2) = \mathbb{Z}\langle 2(a + b), 3b\rangle$. Observing that $\mathbb{Z}\langle a, b\rangle = \mathbb{Z}\langle a + b, b\rangle$, we have

$$\mathsf{H}^{\mathsf{CW}}_1(X) = \frac{\mathsf{ker}(\partial_1)}{\mathsf{im}(\partial_2)} = \mathbb{Z}\langle a + b, b \mid 2(a + b) = 3b = 0\rangle \cong \mathbb{Z}\langle c, b \mid 2c = 3b = 0\rangle \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3.$$

- By the above, $\mathsf{H}_2^{\mathsf{CW}}(X) \cong \mathsf{ker}(\partial_2) = \mathbb{Z}\langle A \rangle \cong \mathbb{Z}$.

Hence $\mathsf{H}_2^{\mathsf{CW}}(X) \cong \begin{cases} \mathbb{Z} & j = 0, \\ \mathbb{Z}_2 \oplus \mathbb{Z}_3 & j = 1, \\ \mathbb{Z} & j = 2, \\ 0 & \text{else.} \end{cases}$ □

(b) Looking at the 2-skeleton of X, we obtain the presentation

$\pi_1(X) = \langle a, b \mid aa^{-1} = 1, b^3 = 1, abab = 1 \rangle = \langle ab, b \mid b^3 = 1, (ab)^2 = 1 \rangle \cong \langle d, b \mid d^2 = b^3 = 1 \rangle$.

The right-hand side is isomorphic to the nonabelian free product $\mathbb{Z}_2 * \mathbb{Z}_3$. □

Problem 4.

Since any embedding is in particular in immersion, the compact 2-manifold $\mathbb{R}\mathsf{P}^2$ can't be embedded into $\mathbb{R}^2$ by problem 1 of 2012, Spring. □

Problem 5.

- Given a vector field $X \in \mathfrak{X}(M)$ and a function $f \in \mathsf{C}^\infty(M)$, we obtain a new function $X(f) \in \mathsf{C}^\infty(M)$ given at each point $x \in M$ by $X(f)(x) := X_x(f)$. In this way, we view X as a map $\mathsf{C}^\infty(M) \to \mathsf{C}^\infty(M)$.
- W.r.t. a local coordinate system $(x_1, \dots, x_m)$ on the m-manifold M, say $X, Y \in \mathfrak{X}(M)$ are written as $X = \sum_{j=1}^m f_j \frac{\partial}{\partial x_j}$ and $Y = \sum_{j=1}^m g_j \frac{\partial}{\partial x_j}$, for some $f_j, g_j \in \mathsf{C}^\infty(M)$, $1 \le j \le m$. Then

$$XY = \sum_{1 \le i,j \le m} f_i \frac{\partial g_j}{\partial x_i} \frac{\partial}{\partial x_j} + \sum_{1 \le i,j \le m} f_i g_j \frac{\partial}{\partial x_i x_j}$$

is a second-order operator (and hence not a vector field) if the second sum is nonzero.

- However,

$$[X, Y] = XY - YX = \sum_{1 \le i,j \le m} f_i \frac{\partial g_j}{\partial x_i} \frac{\partial}{\partial x_j} - \sum_{1 \le i,j \le m} g_j \frac{\partial f_i}{\partial x_j} \frac{\partial}{\partial x_i}$$

is a vector field. Here, the second-order differentials appearing in XY and YX have cancelled by symmetry of mixed partial derivatives.

□

Problem 6.

Note that

$$\int_{\mathsf{S}^3} \omega = \int_{\mathsf{B}^4} \mathsf{d}\omega = \int_{\mathsf{B}^4} (1 + 2w) \mathsf{d}w \wedge \mathsf{d}x \wedge \mathsf{d}y \wedge \mathsf{d}z = \int_{\mathsf{B}^4} \mathsf{d}w \wedge \mathsf{d}x \wedge \mathsf{d}y \wedge \mathsf{d}z + 2 \int_{\mathsf{B}^4} w \mathsf{d}w \wedge \mathsf{d}x \wedge \mathsf{d}y \wedge \mathsf{d}z$$

by Stokes. The second integral on the right vanishes since w is an odd function and B^4 is symmetric about 0. Assuming that $\mathsf{d}x \wedge \mathsf{d}y \wedge \mathsf{d}z \wedge \mathsf{d}w$ is the canonical volume form on $\mathbb{R}^4$, we now have

$$\int_{\mathsf{S}^3} \omega = -\int_{\mathsf{B}^4} \mathsf{d}x \wedge \mathsf{d}y \wedge \mathsf{d}z \wedge \mathsf{d}w = -\mathsf{vol}(\mathsf{B}^4).$$

□

Problem 7.

See problem 3 of 2008, Fall.

2013, Spring

Problem 1.

(a) By Stokes,

$$\int_{\mathsf{S}^2} \omega = \int_{\mathsf{B}^3} \mathsf{d}\omega = \int_{\mathsf{B}^3} (2x+1)\mathsf{d}x \wedge \mathsf{d}y \wedge \mathsf{d}z = 2\int_{\mathsf{B}^3} x\mathsf{d}x \wedge \mathsf{d}y \wedge \mathsf{d}z + \int_{\mathsf{B}^3} \mathsf{d}x \wedge \mathsf{d}y \wedge \mathsf{d}z.$$

The first integral on the right vanishes since x is an odd function and B^3 is symmetric about 0, and so $\int_{\mathsf{S}^2} \omega = \mathsf{vol}(\mathsf{B}^3) = 4\pi/3$. $\square$

(b) If $\alpha \in \Omega^2(\mathbb{R}^3)$ is a closed form with $i^*\alpha = i^*\omega$, then $\int_{\mathsf{S}^2} i^*\alpha = \int_{\mathsf{S}^2} i^*\omega = \int_{\mathsf{S}^2} \omega = 4\pi/3$, but also

$$\int_{\mathsf{S}^2} i^*\alpha = \int_{\mathsf{B}^3} \mathsf{d}(i^*\alpha) = \int_{\mathsf{B}^3} i^*(\mathsf{d}\alpha) = \int_{\mathsf{B}^3} i^*0 = 0,$$

a contradiction. $\square$

Problem 2.

The given functions serve as local coordinates for any point $(x_0, y_0, z_0) \in \mathbb{R}^3$ about which the chart map $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ given by $\phi(x, y, z) := (x, x^2 + y^2 + z^2 - 1, z)$ is a local diffeomorphism. By the inverse function theorem, this condition is equivalent to the following differential being a linear isomorphism,

$$\mathsf{d}\phi_{(x_0,y_0,z_0)} = \begin{pmatrix} 1 & 0 & 0 \\ 2x_0 & 2y_0 & 2z_0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This matrix is invertible exactly when $y_0 \neq 0$, and so ϕ is a coordinate chart on $\mathbb{R}^3 \setminus \{y = 0\}$. $\square$

Problem 3.

We define $\mathbb{R}\mathsf{P}^n$ as the quotient of $\mathbb{R}^{n+1} \setminus 0$ by the relation $x \sim cx$ for all $x \in \mathbb{R}^{n+1} \setminus 0$ and $c \in \mathbb{R} \setminus 0$. Equipping $\mathbb{R}\mathsf{P}^n$ with the quotient topology, it inherits the Hausdorff and second countable properties of $\mathbb{R}^{n+1} \setminus 0$. Now, consider the atlas $\{(U_j, \phi_j)\}_{j=0}^n$ for $\mathbb{R}\mathsf{P}^n$, where for each $1 \leq j \leq n$, we define $U_j := \{x_j \neq 0\} \subset \mathbb{R}\mathsf{P}^n$ and $\phi_j : U_j \to \mathbb{R}^n$,

$$\phi_j([x_0 : \cdots : x_n]) := \left(\frac{x_0}{x_j}, \ldots, \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j}, \ldots, \frac{x_n}{x_j}\right).$$

This chart map is clearly continuous, and has continuous inverse given by

$$\phi_j^{-1}(y_0, \ldots, y_{j-1}, y_{j+1}, \ldots, y_n) := [y_0 : \cdots : y_{j-1} : 1 : y_{j+1} : \cdots : y_n].$$

Moreover if $U_i \cap U_j \neq \varnothing$ for some $1 \leq i, j \leq n$, then the composite $\phi_i \circ \phi_j^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is easily seen to be smooth. Thus $\mathbb{R}\mathsf{P}^n$ is indeed an n-manifold. $\square$

Problem 4.

(a) Let $n \in \mathbb{N}$. Seeing as $\mathsf{H}^1_{\mathsf{dR}}(\mathsf{S}^n) \cong 0$, if $\omega \in \Omega^1(\mathsf{S}^n)$ is closed, then $\omega \in \ker(\mathsf{d}^1) = \mathrm{im}(\mathsf{d}^0)$. $\square$

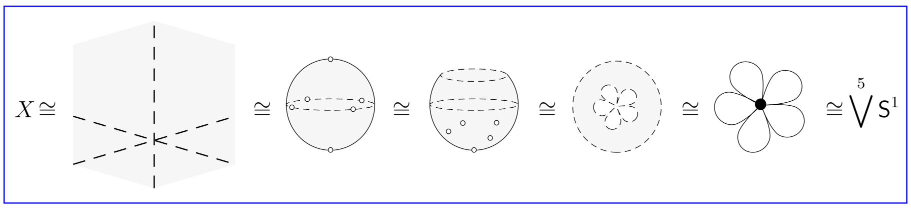
(b) Since $\mathbb{R}\mathsf{P}^n$ is the quotient of S^n by an action of $\mathbb{Z}_2$, we have a canonical projection $\pi : \mathsf{S}^n \twoheadrightarrow \mathbb{R}\mathsf{P}^n$ and a noncanonical inclusion $\iota : \mathbb{R}\mathsf{P}^n \hookrightarrow \mathsf{S}^n$. Let $\omega \in \Omega^1(\mathbb{R}\mathsf{P}^n)$ be closed. Then $\pi^*\omega \in \Omega^1(\mathsf{S}^n)$ is closed since $\mathsf{d}(\pi^*\omega) = \pi^*(\mathsf{d}\omega) = \pi^*0 = 0$. So by part (a), there's some $f \in \Omega^0(S^n)$ with $\mathsf{d}f = \pi^*\omega$. But then

$$\mathsf{d}(\iota^* f) = \iota^*(\mathsf{d}f) = \iota^*(\pi^*\omega) = (\pi \circ \iota)^*\omega = \omega,$$

where $\iota^* f \in \Omega^0(\mathbb{R}\mathsf{P}^n)$, and thus ω is exact. $\square$

Problem 5.

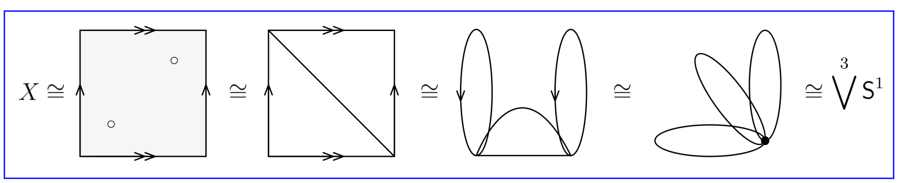
Along each axis, we squish $\mathbb{R}^3$ towards the points lying on the unit sphere, until we're left with a sphere with six points missing (corresponding to the intersections of $\mathbb{R}^3$ with the missing axes). We then stretch out one of these points while moving the remaining missing points close to one another. We're now left with an open disc with five missing points stuck together in the middle. Stretching out each of these missing points and pushing the outside of the disc toward their borders yields a wedge of five circles.



Hence $\pi_1(X) \cong \pi_1\left(\bigvee^5 \mathsf{S}^1\right) \cong \mathsf{F}_5$ and $\mathsf{H}_j(X) \cong \mathsf{H}_j\left(\bigvee^5 \mathsf{S}^1\right) \cong \begin{cases} \mathbb{Z} & j = 0, \\ \mathbb{Z}^{\oplus 5} & j = 1, \\ 0 & \text{else.} \end{cases}$ $\square$

Problem 6.

Viewing the torus as a square with edges identified, upon removing two points, we stretch each missing point out into a triangular region. This leaves the frame of the square together with a diagonal. Identifying the appropriate edges yields a wedge of three circles, as shown.



Then $\mathsf{H}_j(X) \cong \mathsf{H}_j\left(\bigvee^3 \mathsf{S}^1\right) \cong \begin{cases} \mathbb{Z} & j = 0, \\ \mathbb{Z}^{\oplus 3} & j = 1, \\ 0 & \text{else} \end{cases}$ and $\pi_1(X) \cong \pi_1\left(\bigvee^3 \mathsf{S}^1\right) \cong \mathsf{F}_3$, by van Kampen. $\square$

Problem 7.

(a) The 2-sheeted covers of $\mathsf{S}^1 \times \mathsf{S}^1$ are classified by the index-2 subgroups of $\pi_1(\mathsf{S}^1 \times \mathsf{S}^1) \cong \mathbb{Z}^{\oplus 2}$. There are three such covers, corresponding to the subgroups $(2\mathbb{Z}) \oplus \mathbb{Z}, \mathbb{Z} \oplus (2\mathbb{Z})$, and $\mathsf{ker}(f)$ for $f : \mathbb{Z}^{\oplus 2} \to \mathbb{Z}_2$ the homomorphism given by $f(x, y) := [x + y]$. $\square$

(b) Denote by $\pi : \mathbb{R} \twoheadrightarrow \mathsf{S}^1$ the universal cover of S^1, and let $f : X \to \mathsf{S}^1$ be a continuous map. Then we have an induced homomorphism $f_* : \pi_1(X) \to \pi_1(\mathsf{S}^1)$, which must be trivial since $\pi_1(X)$ has torsion (it's finite) while $\pi_1(\mathsf{S}^1) \cong \mathbb{Z}$ doesn't. Then since $\mathbb{R}$ is simply connected, we have $f_*(\pi_1(X)) \cong 1 \subset \pi_*(\pi_1(\mathbb{R})) \cong 1$, and so there exists a lift

$$\begin{tikzcd} & \mathbb{R} \arrow[d, two heads, "\pi"] \\ X \arrow[ur, dashed, "\exists\, \tilde{f}"] \arrow[r, "f"] & \mathsf{S}^1. \end{tikzcd}$$

Again since $\mathbb{R}$ is simply connected, we may choose a homotopy $\{h_t : X \to \mathbb{R}\}_{0 \le t \le 1}$ with $h_0 = \tilde{f}$ and $h_1 = c$ for some constant map $c : X \to \mathbb{R}$. Then $\{\pi \circ h_t\}_{0 \le t \le 1}$ is a homotopy with $\pi \circ h_0 = \pi \circ \tilde{f} = f$, and $\pi \circ h_1 = \pi \circ c$ (a constant map). $\square$

Problem 8.

(a) See problem 3 of 2014, Fall.

(b) Suppose S^{2n} is the universal cover of X, and denote by $G \cong \pi_1(X)$ its group of deck transformations. If $G \cong 1$ then we're done, so assume now that G is nontrivial. The action of G on S^{2n} is free since S^{2n} is path connected; so any $f \in G \setminus 1$ is a homeomorphism $\mathsf{S}^{2n} \to \mathsf{S}^{2n}$ with no fixed points, and $\mathsf{deg}(f) = (-1)^{2n+1} = -1$ by (a). Choosing some $g, h \in G \setminus 1$, we have that

$$\mathsf{deg}(g^2) = [\mathsf{deg}(g)]^2 = (-1)^2 = 1, \quad \mathsf{deg}(gh) = \mathsf{deg}(g)\mathsf{deg}(h) = (-1)(-1) = 1,$$

so $g^2 = 1 = gh$ by the above. This gives $g = h$, and we conclude that G consists of the identity element and a single nontrivial element $g \in G$ with $g^2 = 1$. As such, $G \cong \mathbb{Z}_2$. $\square$

2013, Fall

Problem 1.

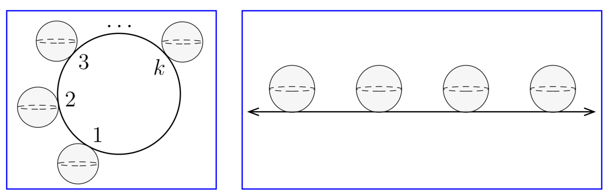
(a) Very similarly to problem 3 of 2006, Spring, we see that $X \cong \mathsf{S}^1 \vee \mathsf{S}^2$. Thus X is obtained by attaching a 1-cell and a 2-cell to a single 0-cell. It's immediate that

$$\mathsf{H}_j(X) \cong \begin{cases} \mathbb{Z} & j = 0,1,2, \\ 0 & \text{else.} \end{cases}$$

$\square$

(b) By van Kampen, $\pi_1(X) \cong \pi_1(\mathsf{S}^1) * \pi_1(\mathsf{S}^2) \cong \mathbb{Z} * 1 \cong \mathbb{Z}$. $\square$

(c) Equivalence classes of connected covers of X are in bijection with the subgroups of $\pi_1(X) \cong \mathbb{Z}$. Any proper subgroup of $\mathbb{Z}$ is of the form $k\mathbb{Z}$ for some $k \in \mathbb{N}$, and corresponds to the k-sheeted cover below on the left. The identity subgroup corresponds to the universal cover on the right.



$\square$

Problem 2.

Since $f : M \to N$ is continuous and M is compact, then $\mathsf{im}(f)$ is compact, and in particular closed. Since f is a submersion, then by the implicit function theorem f is locally an open map (a projection), and hence $\mathsf{im}(f)$ is open. But then $\mathsf{im}(f) \subset N$ is a simultaneously closed and open subset of the connected manifold N, and is nonempty since $M \neq \varnothing$. Thus $\mathsf{im}(f) = N$. $\square$

Problem 3.

See problem 4 of 2010, Fall.

Problem 4.

Recall that we have canonical isomorphisms $\Omega^j(\mathsf{S}^1) \cong \mathsf{C}^\infty(\mathsf{S}^1)$ for $j = 0, 1$.

- We know that S^1 is a 1-manifold, so for all $j \neq 0, 1$, we have $\Omega^j(\mathsf{S}^1) = \varnothing$ and hence $\mathsf{H}^j_{\mathsf{dR}}(\mathsf{S}^1) = 0$.
- Thus
$$\mathsf{H}^0_{\mathsf{dR}}(\mathsf{S}^1) \cong \ker(\mathsf{d}^0) \cong \{f \in \mathsf{C}^\infty(\mathbb{R}) \mid \mathsf{d}f = 0\} \cong \{f \in \mathsf{C}^\infty(\mathbb{R}) \mid f \text{ a constant function}\} \cong \mathbb{R}.$$
- Consider the integration map $I : \Omega^1(\mathsf{S}^1) \to \mathbb{R}$ given by $I(\omega) := \int_{\mathsf{S}^1} \omega$. Choosing $\mathsf{d}t \in \Omega^1(\mathsf{S}^1)$ with $\int_{\mathsf{S}^1} \mathsf{d}t = 2\pi$, then any $c \in \mathbb{R}$ has $c = I((c/2\pi)\mathsf{d}t)$, and so $\mathsf{im}(I) = \mathbb{R}$. Moreover it's easily verified that $\ker(I) = \mathsf{im}(\mathsf{d}^0)$, and so
$$\mathsf{H}^1_{\mathsf{dR}}(\mathsf{S}^1) = \frac{\ker(\mathsf{d}^1)}{\mathsf{im}(\mathsf{d}^0)} = \frac{\Omega^1(\mathsf{S}^1)}{\ker(I)} \cong \mathsf{im}(I) = \mathbb{R}.$$

Hence $\mathsf{H}^j_{\mathsf{dR}}(\mathsf{S}^1) \cong \begin{cases} \mathbb{Z} & j = 0, 1, \\ 0 & \text{else.} \end{cases}$ $\square$

Problem 5.

Background. The quotient Z is called the *double mapping cylinder of f and g*. In this problem we form a long exact sequence which relates the homologies of the constituent spaces X and Y to the homology of Z.

Let $\iota : X \times \partial[0,1] \hookrightarrow X \times [0,1]$ be the canonical inclusion, and let $q : (X \times [0,1], X \times \partial[0,1]) \to (Z, Y)$ be the restriction of the given quotient map $(X \times [0,1]) \coprod Y \twoheadrightarrow Z$ to the first component. Then the exact sequences for the relative homology of the good pairs $(X \times [0,1], X \times \partial[0,1])$ and (Z, Y) give, for each $j \in \mathbb{Z}$, the commutative diagram with exact rows,

$$\begin{tikzcd}
\cdots \arrow[r] & \mathsf{H}_{j+1}(X \times [0,1], X \times \partial[0,1]) \arrow[r, "\delta"] \arrow[d, "q_*"] & \mathsf{H}_j(X \times \partial[0,1]) \arrow[r, "\iota_*"] \arrow[d, "q_*"] & \mathsf{H}_j(X \times [0,1]) \arrow[r] \arrow[d, "q_*"] & \cdots \\
\cdots \arrow[r] & \mathsf{H}_{j+1}(Z, Y) \arrow[r] & \mathsf{H}_j(Y) \arrow[r] & \mathsf{H}_j(Z) \arrow[r] & \cdots .
\end{tikzcd}$$

Thus we're done if we can show that $\mathsf{H}_{j+1}(Z, Y) \cong \mathsf{H}_j(X)$ for each $j \in \mathbb{Z}$. To see this, fix some $j \in \mathbb{Z}$, and note that

$$\mathsf{H}_j(X \times \partial[0,1]) \cong \mathsf{H}_j\left((X \times \{0\}) \coprod (X \times \{1\})\right) \cong \mathsf{H}_j(X)^{\oplus 2}.$$

Both $X \times \{0\}$ and $X \times \{1\}$ are deformation retracts of $X \times [0,1]$, so ι_* is surjective. Then the outer two maps on the top row are 0, and hence δ is injective. As such,

$$\mathsf{H}_{j+1}(X \times [0,1], X \times \partial[0,1]) \cong \mathsf{im}(\delta) \cong \mathsf{ker}(\iota_*) \cong \{(\omega, -\omega) \mid \omega \in \mathsf{H}_j(X)\} \cong \mathsf{H}_j(X),$$

so it's enough to show $\mathsf{H}_{j+1}(Z, Y) \cong \mathsf{H}_{j+1}(X \times [0,1], X \times \partial[0,1])$. Recall that $(X \times [0,1], X \times \partial[0,1])$ and (Z, Y) are good pairs, and q yields induces a homeomorphism $(X \times [0,1])/(X \times \partial[0,1]) \xrightarrow{\sim} Z/Y$. So we may factor the leftmost map q_* as

$$\begin{tikzcd}
\mathsf{H}_{j+1}(X \times [0,1], X \times \partial[0,1]) \arrow[r, "\sim"] \arrow[d, "q_*"] & \tilde{\mathsf{H}}_{j+1}((X \times [0,1])/(X \times \partial[0,1])) \arrow[d, "\sim"] \\
\mathsf{H}_{j+1}(Z, Y) & \tilde{\mathsf{H}}_{j+1}(Z/Y), \arrow[l, "\sim"']
\end{tikzcd}$$

whereby q_* gives the desired isomorphism $\mathsf{H}_{j+1}(X \times [0,1], X \times \partial[0,1]) \xrightarrow{\sim} \mathsf{H}_{j+1}(Z, Y)$. $\square$

Problem 6.

(a) Observe that $\mathbb{Z}_p$ is a finite group, S^3 is Hausdorff, and the given action of $\mathbb{Z}_p$ on S^3 is free; hence this action is properly discontinuous. Then the canonical quotient map $q : \mathsf{S}^3 \twoheadrightarrow L(p, q)$ is a covering map, and

$$\pi_1(L(p,q)) \cong \pi_1(\mathsf{S}^3/\mathbb{Z}_p) \cong \frac{\pi_1(\mathsf{S}^3/\mathbb{Z}_p)}{q_*(\pi_1(\mathsf{S}^3))} \cong \mathbb{Z}_p,$$

since $\pi_1(\mathsf{S}^3) \cong 1$. $\square$

(b) Denote by $\pi : \mathbb{R}^2 \twoheadrightarrow \mathsf{T}^2$ the universal cover of T^2, and let $f : L(p,q) \to \mathsf{T}^2$ be a continuous map. We have an induced map $f_* : \pi_1(L(p,q)) \to \pi_1(\mathsf{T}^2)$, which must be trivial as $\pi_1(L(p,q)) \cong \mathbb{Z}_p$ has torsion while $\pi_1(\mathsf{T}^2) \cong \mathbb{Z}^{\oplus 2}$ doesn't. Then since $\mathbb{R}^2$ is simply connected, we have that $f_*(\pi_1(L(p,q))) \cong 1 \subset \pi_*(\pi_1(\mathbb{R}^2)) \cong 1$, and so there exists a lift

$$\begin{tikzcd} & \mathbb{R}^2 \arrow[d, two heads, "\pi"] \\ L(p,q) \arrow[ur, dashed, "\exists\, \tilde{f}"] \arrow[r, "f"] & \mathsf{T}^2. \end{tikzcd}$$

Again since $\mathbb{R}^2$ is simply connected, we may choose a homotopy $\{h_t : L(p,q) \to \mathbb{R}^2\}_{0\le t\le 1}$ with $h_0 = \tilde{f}$ and $h_1 = c$ for some constant map $c : L(p,q) \to \mathbb{R}^2$. Then $\{\pi \circ h_t\}_{0\le t\le 1}$ is a homotopy which has $\pi \circ h_0 = \pi \circ \tilde{f} = f$, and $\pi \circ h_1 = \pi \circ c$ (a constant map). □

Problem 7.

- For any $a_1, a_2, a_3 \in \mathbb{R}$, denote by $L_{a_1,a_2,a_3} \subset \mathbb{R}^2$ the line determined by the equation

$$a_1 x + a_2 y + a_3 = 0,$$

  and denote by X the space of all lines in $\mathbb{R}^2$. Note that we can't have $a_1 = a_2 = 0$ while $a_3 \neq 0$, and also that $L_{a_1,a_2,a_3} = L_{ca_1,ca_2,ca_3}$ for any $c \in \mathbb{R} \setminus 0$. Thus we have a well defined inclusion map $\iota : X \hookrightarrow \mathbb{R}\mathsf{P}^2 \setminus \{[0:0:1]\}$ given by $\iota(L_{a_1,a_2,a_3}) := [a_1 : a_2 : a_3]$.

- We equip X with the topology induced by this inclusion, whereby X inherits the Hausdorff and second countable properties of $\mathbb{R}\mathsf{P}^2$. Now set $U_1 := \{[1 : a_2 : a_3] \in X\}$ and consider the homeomorphism $\phi_1 : U_1 \to \mathbb{R}^2$ given by $\phi_1([1 : a_2 : a_3]) := (a_2, a_3)$. (Indeed $U_1 \subset \mathbb{R}\mathsf{P}^2$ is open since its complement $\mathbb{R}\mathsf{P}^2 \setminus U_1$ is closed.) Then (U_1, ϕ_1) is a chart for X, and similarly defining $(U_2, \phi_2), (U_3, \phi_3)$, we obtain an atlas $\{(U_j, \phi_j)\}_{j=1}^3$ for X.

- Next, for any $(a_2, a_3) \in \phi_1(U_1 \cap U_2)$, recalling that $a_2 \neq 0$ on U_2, the transition map

$$\tau := \phi_2 \circ \phi_1^{-1}(a_2, a_3) = \phi_2([1 : a_2 : a_3]) = \phi_2([1/a_2 : 1 : a_3/a_2]) = (1/a_2, a_3/a_2)$$

  is smooth. Similarly the other transition maps are also smooth, so this atlas indeed gives X the structure of 2-manifold.

- Finally observe that, with τ as above,

$$\mathsf{det}(\mathsf{d}\tau_{(a_2,a_3)}) = \begin{vmatrix} \dfrac{\partial \tau_1}{\partial a_2} & \dfrac{\partial \tau_1}{\partial a_3} \\ \dfrac{\partial \tau_2}{\partial a_2} & \dfrac{\partial \tau_2}{\partial a_3} \end{vmatrix} = \begin{vmatrix} -a_2^{-2} & 0 \\ -a_3 a_2^{-2} & a_2^{-1} \end{vmatrix} = -a_2^{-3} < 0.$$

  So τ is orientation-reversing, whereby X is unorientable when equipped with this atlas.

□

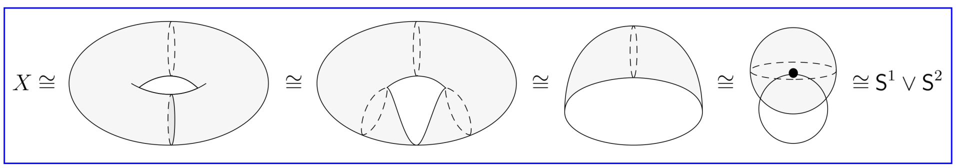
2014, Spring

Problem 1.

No. Similarly to problem 1 of 2005, Fall, we see that $X_1 \cong \mathsf{S}^1$ and $X_2 \cong \mathsf{S}^1 \vee \mathsf{S}^1$. Equivalence classes of connected covers of X_1 are in bijection with the subgroups of $\pi_1(X_1) \cong \pi_1(\mathsf{S}^1) \cong \mathbb{Z}$, and each such subgroup is of the form $k\mathbb{Z}$ for some $k \geq 0$. We know that the identity subgroup corresponds to the simply connected universal cover $\mathbb{R} \twoheadrightarrow \mathsf{S}^1$, and that for any $k \geq 1$, the subgroup $k\mathbb{Z}$ corresponds to the cover $\mathsf{S}^1 \twoheadrightarrow \mathsf{S}^1$ given by $z \mapsto e^{2\pi i/k}z$. Therefore if $X_2 \twoheadrightarrow X_1$ is indeed a (connected) cover, then by the above, X_2 is either simply connected or homeomorphic to S^1. But $\pi_1(X_2) \cong \pi_1(\mathsf{S}^1 \vee \mathsf{S}^1) \cong \mathsf{F}_2$ is nontrivial and not isomorphic to $\pi_1(\mathsf{S}^1) \cong \mathbb{Z}$, so neither of these is a possibility. □

Problem 2.

The space X is created by gluing each point $z \in \partial D$ to a corresponding point $(z, z_0) \in \mathsf{S}^1 \times \mathsf{S}^1$ on the meridianal circle on $\mathsf{S}^1 \times \mathsf{S}^1$ in which the second angular coordinate is fixed at z_0. Since D is contractible, we may shrink it to a point, thereby producing a "croissant." We then transform the shape until we're left with the wedge $\mathsf{S}^1 \vee \mathsf{S}^2$ shown below.



Then immediately $\mathsf{H}_j(X) \cong \mathsf{H}_j(\mathsf{S}^1 \vee \mathsf{S}^2) \cong \begin{cases} \mathbb{Z} & j = 0, 1, 2, \\ 0 & \text{else.} \end{cases}$ □

Problem 3.

See problem 4 of 2007, Fall.

Problem 4.

Yes. Assume that $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a reparametrization $f(x, y) = (s, t)$, where $f_1(x, y) = s$ and $f_2(x, y) = t$ satisfy $X = \frac{\partial}{\partial s} = \mathsf{d}f(\frac{\partial}{\partial x})$ and $Y = \frac{\partial}{\partial t} = \mathsf{d}f(\frac{\partial}{\partial t})$ in some neighborhood of $(0, 1)$. Then

$$2\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} = X = \mathsf{d}f\left(\frac{\partial}{\partial x}\right) = \frac{\partial f_1}{\partial x}\frac{\partial}{\partial x} + \frac{\partial f_2}{\partial x}\frac{\partial}{\partial y}, \qquad \frac{\partial}{\partial y} = Y = \mathsf{d}f\left(\frac{\partial}{\partial y}\right) = \frac{\partial f_1}{\partial y}\frac{\partial}{\partial x} + \frac{\partial f_2}{\partial y}\frac{\partial}{\partial y}$$

and we have the system of equations

$$\frac{\partial f_1}{\partial x} = 2, \quad \frac{\partial f_2}{\partial x} = x, \quad \frac{\partial f_1}{\partial y} = 0, \quad \frac{\partial f_2}{\partial y} = 1.$$

Solving this yields $f_1(x, y) = 2x + c_1$ and $f_2(x, y) = \frac{1}{2}x^2 + y + c_2$ for some $c_1, c_2 \in \mathbb{R}$. If the (soon-to-be) local coordinate system given by f is centered at $(0, 1)$, then

$$(0, 0) = f(0, 1) = \left(2x + c_1, \frac{1}{2}x^2 + y + c_2\right)\Big|_{(0,1)} = (c_1, 1 + c_2) \implies c_1 = 0, \quad c_2 = -1,$$

and thus we need $f_1(x,y) = 2x$ and $f_2(x,y) = \frac{1}{2}x^2 + y - 1$. And now, by the inverse function theorem since, f does indeed provide a local coordinate system about $(0,1)$ since

$$\mathsf{d}f_{(0,1)} = \begin{pmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \end{pmatrix}\Bigg|_{(0,1)} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

is invertible. □

Problem 5.

See problem 6 of 2005, Fall.

Problem 6.

Since $x^2 + y^2 + z^2 = 1$ on S^2, then by a simple calculation $\mathsf{d}\omega = 3\mathsf{d}x \wedge \mathsf{d}y \wedge \mathsf{d}z$ on S^2, whereby

$$\int_{\mathsf{S}^2} \omega = \int_{\mathsf{B}^3} \mathsf{d}\omega = 3\int_{\mathsf{B}^3} \mathsf{d}x \wedge \mathsf{d}y \wedge \mathsf{d}z = 3\mathsf{vol}(\mathsf{B}^3) = 4\pi$$

by Stokes. □

Problem 7.

Remark. There's a mistake in the problem statement. We wish to show that the space of points $x \in \mathbb{R}^m$ such that $M \cap (\{x\} \times \mathbb{R}^n)$ is infinite has measure 0.

Let $\iota : M \hookrightarrow \mathbb{R}^m \times \mathbb{R}^n$ and $\pi : \mathbb{R}^m \times \mathbb{R}^n \twoheadrightarrow \mathbb{R}^m$ be the canonical inclusion and projection maps, respectively, and let $f := \pi \circ \iota : M \to \mathbb{R}^m$.

$$\begin{tikzcd} M \arrow[r, hook, "\iota"] \arrow[dr, "f"'] & \mathbb{R}^m \times \mathbb{R}^n \arrow[d, two heads, "\pi"] \\ & \mathbb{R}^m \end{tikzcd}$$

Let $x \in \mathbb{R}^m$ be a regular value of f. Then for any $y \in f^{-1}(x)$, the map $\mathsf{d}f_y : \mathsf{T}_yM \to \mathsf{T}_x\mathbb{R}^m$ is a surjective linear map of m-dimensional vector spaces, and thus a linear isomorphism. So by the inverse function theorem, there's an open neighborhood $U_y \subset M$ of y such that $f\big|_{U_y} : U_y \to f(U_y)$ is a diffeomorphism. Now, $f^{-1}(x)$ is a closed subset of the compact manifold M, since $\{x\} \subset \mathbb{R}^m$ is closed, and thus $f^{-1}(x)$ is itself compact. Then the open cover $\{U_y\}_{y \in f^{-1}(x)}$ of $f^{-1}(x)$ admits a finite subcover $\{U_{y_j}\}_{j=1}^k$. If $y \in f^{-1}(x)$ belongs to U_{y_j} for some $1 \le j \le k$, then we have $f\big|_{U_{y_j}}(y) = x = f\big|_{U_{y_j}}(y_j)$, and so $y = y_j$ since $f\big|_{U_{y_j}}$ is a diffeomorphism. Thus U_{y_j} contains no more than one element of $f^{-1}(x)$, for each $1 \le j \le k$, and since $\{U_{y_j}\}_{j=1}^k$ is a cover of $f^{-1}(x)$, it follows that $f^{-1}(x)$ is finite. Then

$$\begin{aligned} f^{-1}(x) &= \{(y_1,y_2) \in M \subset \mathbb{R}^m \times \mathbb{R}^n \mid f(y_1,y_2) = x\} = M \cap \{(y_1,y_2) \in \mathbb{R}^m \times \mathbb{R}^n \mid \pi(y_1,y_2) = x\} \\ &= M \cap \{(y_1,y_2) \in \mathbb{R}^m \times \mathbb{R}^n \mid y_1 = x\} = M \cap (\{x\} \times \mathbb{R}^n) \end{aligned}$$

is finite. So if $x \in \mathbb{R}^m$ is such that $M \cap (\{x\} \times \mathbb{R}^n)$ is infinite, then x is a critical value of f. By Sard, the critical values of f have measure 0. □

2014, Fall

Problem 1.

Let $p : (\tilde{X}, \tilde{x}) \twoheadrightarrow (X, x)$ be the (compact) universal cover of X, where $\tilde{x} \in \tilde{X}$ is some point in the fiber $p^{-1}(x) \subset \tilde{X}$. We have a bijection $\pi_1(X, x) \to p^{-1}(x)$ given by associating to a loop $[f] \in \pi_1(X, x)$ the point $\tilde{f}(1) \in p^{-1}(x)$, where $\tilde{f} : [0, 1] \to \tilde{X}$ is the unique lift (of a representative $f : [0, 1] \to X$) satisfying $\tilde{f}(0) = \tilde{x}$. But $p^{-1}(x)$ is finite since it's a discrete closed subset of the compact space $\tilde{X}$, and so $\pi_1(X, x)$ must be finite as well. $\square$

Problem 2.

We have that

$$\mathsf{H}_j(\mathsf{S}^1 \vee \mathsf{S}^1 \vee \mathsf{S}^2) \cong \mathsf{H}_j(\mathsf{S}^1) \oplus \mathsf{H}_j(\mathsf{S}^1) \oplus \mathsf{H}_j(\mathsf{S}^2) \cong \begin{cases} \mathbb{Z} & j = 0, \\ \mathbb{Z}^{\oplus 2} & j = 1, \\ \mathbb{Z} & j = 2, \\ 0 & \text{else} \end{cases} \cong \mathsf{H}_j(\mathsf{T}^2).$$

Yet, $\pi_1(\mathsf{S}^1 \vee \mathsf{S}^1 \vee \mathsf{S}^2) \cong \mathbb{Z} * \mathbb{Z}$ is nonabelian, while $\pi_1(\mathsf{T}^2) \cong \mathbb{Z}^{\oplus 2}$ is abelian, so $\mathsf{S}^1 \vee \mathsf{S}^1 \vee \mathsf{S}^2$ and T^2 can't be homeomorphic. $\square$

Problem 3.

Let $a : \mathsf{S}^n \to \mathsf{S}^n$ be the antipodal map given by $a(x) := -x$.

(i) Suppose $f : \mathsf{S}^n \to \mathsf{S}^n$ has no fixed points. Since $\mathsf{S}^n \subset \mathbb{R}^{n+1}$, we may use the vector space structure of $\mathbb{R}^{n+1}$ to define a family of maps $\{h_t : \mathsf{S}^n \to \mathsf{S}^n\}_{0 \le t \le 1}$ by

$$h_t(x) := \frac{(1-t)f(x) - tx}{\|(1-t)f(x) - tx\|}.$$

Clearly the denominator is nonzero when $t = 1$ since $x \in \mathsf{S}^n$. Now assume that for some $0 \le t_0 < 1$, the denominator is 0; then $f(x) = \frac{t_0}{1-t_0}x$. Since $\|x\| = \|f(x)\| = 1$, we must have $t_0 = 1/2$. But then $f(x) = x$, a contradiction. Thus the denominator is always nonzero on S^n, whereby $\{h_t\}_{0 \le t \le 1}$ is a well defined homotopy between $h_0 = f$ and $h_1 = a$. $\square$

(ii) With f as above, assume that there are no points $x \in \mathsf{S}^{2m}$ such that $f(x) = x$ or $f(x) = -x$. Then both $-f, f : \mathsf{S}^{2m} \to \mathsf{S}^{2m}$ are free of fixed points, and so by (a) are homotopic to a. Therefore we have $\mathsf{deg}(f) = \mathsf{deg}(a) = (-1)^{2m+1} = -1$, but on the other hand

$$\mathsf{deg}(f) = \mathsf{deg}(a \circ (-f)) = \mathsf{deg}(a)\mathsf{deg}(-f) = (-1)^{2m+1}\mathsf{deg}(-f) = -\mathsf{deg}(a) = -(-1)^{2m+1} = 1,$$

a contradiction. $\square$

Problem 4.

See problem 5 of 2005, Fall, and problem 4 of 2013, Fall.

Problem 5.

We're given that X is homeomorphic to a genus-g surface, and that $\partial X = \varnothing$. So by Gauss-Bonnet,

$$\iint_X K \mathsf{d}A = 2\pi\chi(X) = 2\pi(2 - 2g) < 0$$

since $g > 1$. It follows that $K < 0$ on a subset $U \subset M$ with nonempty interior. Choosing an interior point $x \in U$ and an open neighborhood $V \subset U$ of x, we have that $K < 0$ on V. $\square$

Problem 6.

Since $\omega \in \Omega^d(M)$ is a volume form, then $\mathsf{d}\omega = 0$. Hence by Cartan and Stokes,

$$\int_M \mathcal{L}_X \omega = \int_M (\mathsf{d}\iota_X \omega + \iota_X \underbrace{\mathsf{d}\omega}_{=0}) = \int_{\partial M} \iota_X \omega = 0,$$

because $\partial M = \varnothing$. This implies that $\mathcal{L}_X \omega$ must vanish at some point of M. $\square$

Problem 7.

See problem 3 of 2013, Spring, replacing real numbers by complex ones.

2015, Fall

Problem 1.

(a)
- A *homotopy* between two continuous maps $f, g : X \to Y$ of topological spaces is a continuous map $h : X \times [0,1] \to Y$ with $h(\cdot, 0) = f$ and $h(\cdot, 1) = g$. In this case, f and g are said to be *homotopic*.
- Two topological spaces X, Y are said to be *homotopy equivalent* if there exists a pair of continuous maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ is homotopic to id_X, and $f \circ g$ is homotopic to id_Y. In this case, f and g are called *homotopy equivalences* between X and Y.

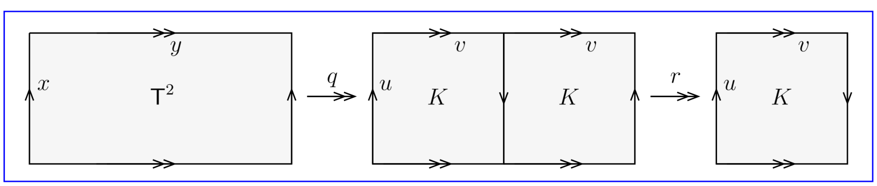
(b) The closed disc B^2 is homotopy equivalent to a point $*$ (since it's contractible), but B^2 and $*$ aren't homeomorphic since any map $\mathsf{B}^2 \to *$ is noninjective. □

(c) Both the sphere S^2 and the point $*$ have trivial fundamental group, but aren't homotopy equivalent since $*$ is contractible while S^2 isn't. □

(d) The torus T^2 and the wedge of two circles $\mathsf{S}^1 \vee \mathsf{S}^1$ both have first homology group isomorphic to $\mathbb{Z}^{\oplus 2}$, but the fundamental group of T^2 is the abelian group $\mathbb{Z}^{\oplus 2}$ while that of $\mathsf{S}^1 \vee \mathsf{S}^1$ is the nonabelian free group F_2. □

Problem 2.

(a) Let $p : \mathsf{T}^2 \twoheadrightarrow K$ be the composite of the quotient map q from T^2 to two Klein bottles K glued to one another as shown, followed by a projection r from this space onto a single copy of K.



This is the desired cover. □

(b) Let $x, y \in \pi_1(\mathsf{T}^2)$ and $u, v \in \pi_1(K)$ be loops in T^2 and K, respectively, corresponding to the edges above as labeled. We see that

$$\pi_1(\mathsf{T}^2) \cong \langle x, y \mid xyx^{-1}y^{-1} = 1 \rangle, \quad \pi_1(K) \cong \langle u, v \mid uvuv^{-1} = 1 \rangle.$$

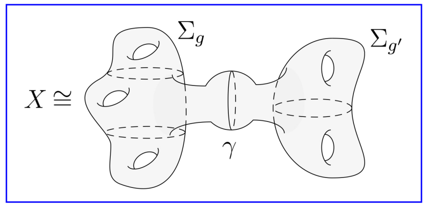
Moreover, by the diagram, $p_*(x) = u$ and $p_*(y) = r_*(q_*(y)) = r_*(2v) = 2r_*(v) = 2v$. □

Problem 3.

(a) Recall first that

$$\pi_1(\Sigma_g) \cong \left\langle x_i, y_i, 1 \le i \le g \,\middle|\, \prod_{i=1}^{g} [x_i, y_i] = 1 \right\rangle, \quad \pi_1(\Sigma_{g'}) \cong \left\langle u_j, v_j, 1 \le i \le g' \,\middle|\, \prod_{j=1}^{g'} [u_j, v_j] = 1 \right\rangle.$$

Within X, the copies of Σ_g and $\Sigma_{g'}$ intersect along the circular curve $\gamma = \gamma'$.



The inclusion $\gamma \hookrightarrow \Sigma_g$ induces the trivial homomorphism $\pi_1(\gamma) \to \pi_1(\Sigma_g)$ since, in Σ_g, the curve γ forms the boundary of an embedded (contractible) disc. By similar reasoning the inclusion $\gamma \hookrightarrow \Sigma_{g'}$ induces the trivial homomorphism $\pi_1(\gamma) \to \pi_1(\Sigma_{g'})$ as well, so by van Kampen

$$\pi_1(X) \cong \pi_1(\Sigma_g) * \pi_1(\Sigma_{g'}) \cong \left\langle x_i, y_i, u_j, v_j, 1 \le i \le g, 1 \le j \le g' \,\middle|\, \prod_{i=1}^{g}[x_i, y_i] = \prod_{j=1}^{g'}[u_j, v_j] = 1 \right\rangle.$$

□

(b) We already have by path connectedness that $\mathsf{H}_0(X) \cong \mathbb{Z}$, and by Hurewicz that

$$\mathsf{H}_1(X) \cong \pi_1(X)/[\pi_1(X), \pi_1(X)] \cong \mathbb{Z}^{\oplus 2g} \oplus \mathbb{Z}^{\oplus 2g'} \cong \mathbb{Z}^{\oplus 2(g+g')}.$$

Letting U and V be, respectively, Σ_g and $\Sigma_{g'}$ extended slightly beyond γ within X, then $U \cong \Sigma_g, V \cong \Sigma_{g'}$, and $U \cap V \cong \mathsf{S}^1$. For any $j \ge 2$, Mayer-Vietoris immediately yields $\mathsf{H}_j(X) \cong 0$. We further have by Mayer-Vietoris the exact sequence

$$0 \longrightarrow \mathbb{Z}^{\oplus 2} \xrightarrow{f} \mathsf{H}_2(X) \xrightarrow{\partial} \mathbb{Z} \xrightarrow{\iota} \mathbb{Z}^{\oplus 2(g+g')}.$$

By exactness, f is injective, whereby $\mathsf{ker}(\partial) \cong \mathsf{im}(f) \cong \mathbb{Z}^{\oplus 2}$. Moreover, as observed before, the map ι induced by the inclusions of γ into Σ_g and $\Sigma_{g'}$ is trivial, and so $\mathsf{im}(\partial) \cong \mathsf{ker}(\iota) \cong \mathbb{Z}$. Thus $\mathsf{H}_2(X) \cong \mathbb{Z}^{\oplus 3}$, and in summary

$$\mathsf{H}_j(X) \cong \begin{cases} \mathbb{Z} & j = 0, \\ \mathbb{Z}^{\oplus 2(g+g')} & j = 1, \\ \mathbb{Z}^{\oplus 3} & j = 2, \\ 0 & \text{else.} \end{cases}$$

□

(c) **No**. We have $\mathsf{H}_2(\Sigma_g \times \Sigma_{g'}) \cong \mathsf{H}_2(\Sigma_g) \oplus \mathsf{H}_2(\Sigma_{g'}) \cong \mathbb{Z}^{\oplus 2}$, but we just showed that $\mathsf{H}_2(X) \cong \mathbb{Z}^{\oplus 3}$. Therefore $\Sigma_g \times \Sigma_{g'}$ and X can't be homotopy equivalent. □

Problem 4.

Assume $\mathsf{d}\omega \neq 0$. Then there's a point $x \in M$ at which $\mathsf{d}\omega_x \neq 0$. Let $U \subset M$ be a neighborhood of x homeomorphic to $\mathsf{B}^n \subset \mathbb{R}^n$ via some coordinate chart $\phi : U \to \mathsf{B}^n$ with local U-coordinates $y_1, \dots, y_n$. Then on U, we may write $\mathsf{d}\omega$ in the local form $\mathsf{d}\omega = f\mathsf{d}y_1 \wedge \dots \wedge \mathsf{d}y_n$, for some

$f \in \mathsf{C}^\infty(U)$. Since $f(x)$ is nonzero, say w.l.o.g. $f(x) > 0$, then also w.l.o.g. U was chosen small enough so that $f > 0$ on all of U by continuity of f. Then

$$\int_U \mathsf{d}\omega = \int_{\mathsf{B}^n} (\phi^{-1})^*(\mathsf{d}\omega) = \int_{\mathsf{B}^n} \underbrace{(f \circ \phi^{-1})}_{>0} \mathsf{d}z_1 \wedge \ldots \wedge \mathsf{d}z_n > 0,$$

where $z_j =: \phi^* y_j$ is the B^n-coordinate corresponding to y_j, for each $1 \leq j \leq n$. But on the other hand, $\partial U \subset M$ is an oriented closed submanifold since it's homeomorphic to $\partial \mathsf{B}^n = \mathsf{S}^{n-1}$ via ϕ, so by Stokes and the problem assumption, $\int_U \mathsf{d}\omega = \int_{\partial U} \omega = 0$, a contradiction. □

Problem 5 (?)**.**

By Frobenius, it's enough to verify that $[v, w] = 0$. Now,

$$vw = \left(\frac{\partial}{\partial x} + xz\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial y} + yz\frac{\partial}{\partial z}\right) = \frac{\partial^2}{\partial x \partial y} + xz\frac{\partial^2}{\partial y \partial z} + 0 + yz\frac{\partial^2}{\partial x \partial z} + xyz\frac{\partial}{\partial z} + xyz^2\frac{\partial^2}{\partial z^2},$$
$$wv = \left(\frac{\partial}{\partial y} + yz\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x} + xz\frac{\partial}{\partial z}\right) = \frac{\partial^2}{\partial x \partial y} + yz\frac{\partial^2}{\partial x \partial z} + 0 + xz\frac{\partial^2}{\partial y \partial z} + xyz\frac{\partial}{\partial z} + xyz^2\frac{\partial^2}{\partial z^2},$$

whereby $[v, w] = vw - wv = 0$. □

Problem 6.

Remark. In this problem, we use $\mathbb{C} \cup \{\infty\}$ and S^2 interchangeably by implicitly making use of the given homeomorphism. Note also that if $f : \mathbb{C} \to \mathbb{C}$ is a constant polynomial, then it trivially extends to S^2, and this extension has topological degree 0, which is the same as the algebraic degree of f. So we'll also assume w.l.o.g. that f is nonconstant.

(a) Define $\bar{f} : \mathsf{S}^2 \to \mathsf{S}^2$ by setting $\bar{f}\big|_{\mathbb{C}} := f$ and $\bar{f}(\infty) := \infty$. Clearly $\bar{f}$ is continuous on $\mathbb{C}$, so it remains to check continuity at ∞. Indeed, if $\{z_j\}_{j=1}^\infty \subset \mathsf{S}^2$ is a sequence converging to ∞, then

$$\lim_{j\to\infty} \bar{f}(z_j) = \lim_{j\to\infty} f(z_j) = \infty = \bar{f}(\infty),$$

where we have the third equality, by Liouville, since f is a nonconstant polynomial. □

(b) Say $f(z) = a_0 + a_1 z + \cdots + a_m z^m$ for all $z \in \mathbb{C}$, where $a_0, \ldots, a_m \in \mathbb{C}$ and $a_m \neq 0$. Then the algebraic degree of f is $m \in \mathbb{N}$, and it's enough to show that $\bar{f}$ is homotopic to the map $g : \mathsf{S}^2 \to \mathsf{S}^2$ given by $g(z) := z^m$ for all $z \in \mathsf{S}^2$, since g has homological degree m. We begin with the map

$$h : \mathsf{S}^2 \times [0,1] \to \mathsf{S}^2, \quad h(z,t) := t(a_0 + a_1 z + \cdots + a_{m-1} z^{m-1}) + a_m z^m,$$

with $h_0(z) = a_m z^m$ for all $z \in \mathsf{S}^2$, and $h_1 = f$. Obviously h is continuous on $\mathbb{C} \times [0,1]$, so it remains to check that it's also continuous at any point of the form $(\infty, t) \in \mathsf{S}^2 \times [0,1]$.

Take any $(\infty, t) \in \mathsf{S}^2 \times [0,1]$ and any $M > 0$. We need to check that there's some $K > 0$ large enough and $\delta > 0$ small enough so that whenever $(z, s) \in \mathsf{S}^2 \times [0,1]$ has $|z| > K$ and $|s - t| < \delta$, then $|h(z,s)| > M$. But indeed, $|a_m z^m| > \left|a_0 + a_1 z + \cdots + a_{m-1} z^{m-1}\right|$ whenever $|z| > K$ for some large $K > 0$, and so choosing this value of K together with $\delta := 1$ proves the desired continuity. Therefore h is a homotopy between f and h_0. Similarly, we can check that the map

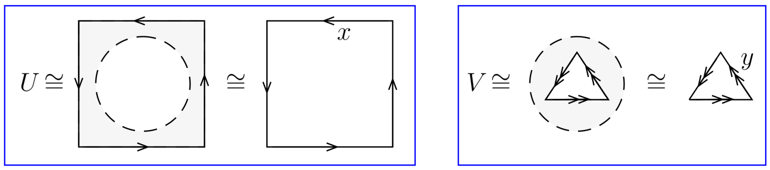
$$k : \mathsf{S}^2 \times [0,1], \quad k(z,t) := a_m^t z^m$$

is a homotopy between h_0 and g, and this completes the proof. □

2016, Spring

Problem 1.

We write X as the union of the subspaces U and V shown below, with $U \cap V \cong \mathsf{S}^1$.



Let $x \in \pi_1(U)$ and $y \in \pi_1(V)$ correspond to the edges above as labeled. Letting $i : U \cap V \hookrightarrow U$ and $j : U \cap V \hookrightarrow V$ be the canonical inclusions, then the induced homomorphism $i_* : \pi_1(U \cap V) \to \pi_1(U)$ maps the single generator $1 \in \pi_1(U \cap V) \cong \pi_1(\mathsf{S}^1) \cong \mathbb{Z}$ to $i_*(1) = x^4$, and $j_* : \pi_1(U \cap V) \to \pi_1(V)$ maps it to $j_*(1) = y^3$. So by van Kampen,

$$\pi_1(X) \cong \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V) \cong \frac{\langle x, y \rangle}{\langle x^4 y^{-3} \rangle} = \big\langle x, y \mid x^4 = y^3 = 1 \big\rangle.$$

□

Problem 2.

- Letting $\mathsf{Bij}(p^{-1}(x_0))$ denote the set of bijections $p^{-1}(x_0) \to p^{-1}(x_0)$, we have an assignment

  $$F : \pi_1(X, x_0) \to \mathsf{Bij}(p^{-1}(x_0)), \quad F_{[\gamma]}(\tilde{x}) := \tilde{\gamma}_{\tilde{x}}(1),$$

  where $\tilde{\gamma}_{\tilde{x}} : [0,1] \to \tilde{X}$ is the unique lift of γ satisfying $\tilde{\gamma}_{\tilde{x}}(0) = \tilde{x}$. This assignment is precisely the monodromy action of $\pi_1(X, x_0)$ on $p^{-1}(x_0)$, and as such, for any $[\gamma] \in \pi_1(X, x_0)$, the order of $F_{[\gamma]}$ must divide $|\pi_1(X, x_0)| = |\mathbb{Z}_5| = 5$.

- Take some connected component $\tilde{X}_1 \subset \tilde{X}$, and a point $\tilde{x} \in \tilde{X}_1$. There exists a path in X from $\pi(\tilde{x})$ to x_0, and this path lifts to a path in $\tilde{X}_1$ from $\tilde{x}_0$ to an element of $p^{-1}(x_0)$. Since $\tilde{X}_1$ is connected, this means that this element of $p^{-1}(x_0)$ belongs to $\tilde{X}_1$.

  Next suppose a connected component $\tilde{X}_1 \subset \tilde{X}$ contains a distinct pair $\tilde{x}_0, \tilde{x}_1 \in p^{-1}(x_0)$. There exists a path $\tilde{\gamma}$ in $\tilde{X}_1$ from $\tilde{x}_0$ to $\tilde{x}_1$. Then $\pi \circ \tilde{\gamma}$ is a loop in X based at x_0, and $F_{[\pi \circ \tilde{\gamma}]}$ is nontrivial (it for instance sends $\tilde{x}_0$ to $\tilde{x}_1$).

  Thus if $F_{[\gamma]}$ has order 1 for each $[\gamma] \in \pi_1(X, x_0)$ (i.e. if the action F is trivial), then no connected component of $\tilde{X}$ contains more than one element of $p^{-1}(x_0)$. But we also showed that each connected component of $\tilde{X}$ contains at least one such element, and we conclude that $\tilde{X}$ has exactly one connected component for each element of $p^{-1}(x_0)$. So $\tilde{X}$ has six connected components.

- Finally, if F is nontrivial, then there's some $[\gamma] \in \pi_1(X, x_0)$ such that $F_{[\gamma]}$ has order 5, and as such, there's some connected component $\tilde{X}_1$ of $\tilde{X}$ containing at least 5 elements of $p^{-1}(x_0)$. The elements of $p^{-1}(x_0)$ contained in $\tilde{X}_1$ belong to a single orbit of the action F since $\tilde{X}_1$ is connected, and the order of any such orbit must divide $|\pi_1(X, x_0)| = 5$. Hence there are exactly 5 elements of $p^{-1}(x_0)$ contained in $\tilde{X}_1$, and the last element belongs to some other connected component of $\tilde{X}$. So $\tilde{X}$ has two connected components.

□

Problem 3.

By problem 5 of 2005, Fall, we have $\mathsf{H}_j(\mathsf{S}^n \times \mathsf{S}^1) \cong \mathsf{H}_j(\mathsf{S}^n) \oplus \mathsf{H}_{j-1}(\mathsf{S}^n)$ for all $j \in \mathbb{Z}$. Thus

$$\mathsf{H}_j(\mathsf{S}^1 \times \mathsf{S}^1) \cong \left(\begin{cases} \mathbb{Z} & j = 0, 1, \\ 0 & \text{else} \end{cases}\right) \oplus \left(\begin{cases} \mathbb{Z} & j = 1, 2, \\ 0 & \text{else} \end{cases}\right) \cong \begin{cases} \mathbb{Z} & j = 0, \\ \mathbb{Z}^{\oplus 2} & j = 1, \\ \mathbb{Z} & j = 2, \\ 0 & \text{else,} \end{cases}$$

and for $n \geq 2$,

$$\mathsf{H}_j(\mathsf{S}^n \times \mathsf{S}^1) \cong \left(\begin{cases} \mathbb{Z} & j = 0, n, \\ 0 & \text{else} \end{cases}\right) \oplus \left(\begin{cases} \mathbb{Z} & j = 1, n+1, \\ 0 & \text{else} \end{cases}\right) \cong \begin{cases} \mathbb{Z} & j = 0, 1, n, n+1, \\ 0 & \text{else.} \end{cases}$$

□

Problem 4.

(a) Since $\mathsf{im}(f)$ has nonempty interior, it has positive Lebesgue measure, so by Sard we may choose a regular value $y \in \mathsf{im}(f)$ of f. Now take some $x \in f^{-1}(y)$. Then $\mathsf{d}f_x : \mathsf{T}_x M \to \mathsf{T}_y \mathbb{R}^n$ is a surjective linear map of n-dimensional vector spaces, and thereby a linear isomorphism. So by the inverse function theorem, there exists an open neighborhood $U \subset M$ of x such that $f\big|_U : U \to f(U)$ is a diffeomorphism. □

(b) Since M is compact and f is continuous, $\mathsf{im}(f)$ is compact, and in particular not all of $\mathbb{R}^n$. So f isn't surjective, and $\mathsf{deg}(f) = 0$. Let $y \in \mathsf{im}(f)$ be as in part (a). Then

$$0 = \mathsf{deg}(f) = \sum_{x \in f^{-1}(y)} \mathsf{deg}_x(f).$$

Recall that $f^{-1}(y) \neq \varnothing$, and each local degree $\mathsf{deg}_x(f) = \pm 1$. So to get zero on the left-hand side, there must be points $x_1, x_2 \in f^{-1}(y)$ so that $\mathsf{deg}_{x_1}(f) = 1$ and $\mathsf{deg}_{x_2}(f) = -1$. Then f is orientation-preserving at x_1 and orientation-reversing at x_2. □

Problem 5.

Since $\mathbb{R}\mathsf{P}^n$ is an n-manifold, then there exists a nowhere vanishing volume form $\omega \in \Omega^n(\mathbb{R}\mathsf{P}^n)$ if and only if $\mathbb{R}\mathsf{P}^n$ is orientable. And by problem 3 of 2011, Spring, $\mathbb{R}\mathsf{P}^n$ is orientable if and only if either $n \geq 0$ is odd or (trivially) $n = 0$. □

Problem 6.

See problem 4 of 2008, Spring.

2017, Spring

Problem 1.

Background. A *symplectic manifold* is a pair (M^{2n}, ω) consisting of an even-dimensional manifold M together with a closed nondegenerate 2-form $\omega \in \Omega^2(M)$. It follows from what we prove in this problem that an *exact symplectic manifold*, that is, a symplectic manifold (M, ω) with ω exact, also has exact *symplectic volume form* $\omega^{\wedge n}$.

Since $\mathsf{d}\omega = 0$, we have $\mathsf{d}(\alpha \wedge \underbrace{\omega \wedge \ldots \wedge \omega}_{(n-1) \text{ times}}) = (\mathsf{d}\alpha) \wedge \underbrace{\omega \wedge \ldots \wedge \omega}_{(n-1) \text{ times}} = \underbrace{\omega \wedge \ldots \wedge \omega}_{n \text{ times}}$. □

Problem 2.

The 3-sphere

$$\mathsf{S}^3 = \{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 2\}$$

may be written as the union of the two solid tori

$$U := \{(z, w) \in \mathsf{S}^3 \mid |z|^2 \geq 1\} = \{(z, w) \in \mathsf{S}^3 \mid |w|^2 \leq 1\} \cong \mathsf{S}^1 \times \mathsf{B}^2,$$
$$V := \{(z, w) \in \mathsf{S}^3 \mid |z|^2 \leq 1\} = \{(z, w) \in \mathsf{S}^3 \mid |w|^2 \geq 1\} \cong \mathsf{B}^2 \times \mathsf{S}^1,$$

glued along the common boundary

$$\partial U = \partial V = \{(z, w) \in \mathsf{S}^3 \mid |z|^2 = |w|^2 = 1\} \cong \mathsf{S}^1 \times \mathsf{S}^1.$$

Thus $X \cong \mathsf{S}^3$, whereby $\pi_1(X) \cong \pi_1(\mathsf{S}^3) \cong 1$, since any n-sphere with $n \geq 2$ is simply connected. □

Problem 3.

By the above, $\mathsf{H}_j(X) \cong \mathsf{H}_j(\mathsf{S}^3) \cong \begin{cases} \mathbb{Z} & j = 0, 3, \\ 0 & \text{else.} \end{cases}$ □

Problem 4.

Background. In this problem we prove a form of Whitney's embedding theorem.

Fix some $v \in \mathsf{S}^{n-1}$, and let $x, y \in M$ with $x \neq y$. Then $\pi_v(x) = \pi_v(y) \iff x - y = cv$ for some $c \in \mathbb{R} \iff (x-y)/\|x-y\| = v$. So we see that the restriction $\pi_v\big|_M$ is injective if and only if v is not in the image of the smooth map $f : (M \times M) \setminus \Delta_M \to \mathsf{S}^{n-1}$ given by $f(x, y) := (x-y)/\|x-y\|$, where $\Delta_M := \{(x, x) \in M \times M\}$. In other words, $\pi_v\big|_M$ is injective for all $v \in \mathsf{S}^{n-1} \setminus \mathsf{im}(f)$, so it remains to check that $\mathsf{im}(f)$ has measure 0. But this holds by a corollary of Sard since the dimension of the domain is strictly less than that of the codomain,

$$\mathsf{dim}_{\mathbb{R}}((M \times M) \setminus \Delta_M) = 2 \cdot \mathsf{dim}_{\mathbb{R}}(M) \leq 2\left(\frac{n}{2} - 1\right) = n - 2 < n - 1 = \mathsf{dim}_{\mathbb{R}}(\mathsf{S}^{n-1}).$$

□

Problem 5.

See problem 5 of 2011, Spring.

Problem 6.

See problem 7 of 2007, Fall.

2017, Fall

Problem 1.

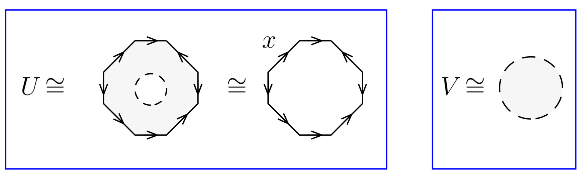
Since M is compact and f is continuous, $\mathsf{im}(f)$ is compact, and in particular not all of $\mathbb{R}^m$. So f isn't surjective, and $\mathsf{deg}(f) = 0$. Let $y \in \mathbb{R}^m$ be a regular value of f; by Sard, such points have full measure in $\mathbb{R}^m$. We have

$$0 = \mathsf{deg}(f) = \sum_{x \in f^{-1}(y)} \mathsf{deg}_x(f).$$

But each local degree $\mathsf{deg}_x(f) = \pm 1$, so to obtain 0 on the left-hand side, there must be an even number of points belonging to $f^{-1}(y)$. □

Problem 2.

Let X be the given quotient space, and write X as the union of the subspaces U and V shown below, with $U \cap V \cong \mathsf{S}^1$.



Let $x \in \pi_1(U)$ correspond to the edge above as labeled. Observe that $\pi_1(V) \cong 1$ since V is contractible, and that $\pi_1(U) \cong \mathbb{Z}$, generated by the single element x. Letting $i : U \cap V \hookrightarrow U$ and $j : U \cap V \hookrightarrow V$ be the canonical inclusions, then the induced homomorphism $i_* : \pi_1(U \cap V) \to \pi_1(U)$ maps the single generator $1 \in \pi_1(U \cap V) \cong \pi_1(\mathsf{S}^1) \cong \mathbb{Z}$ to $i_*(1) = xxx^{-1}xx^{-1}x^{-1}x^{-1}x^{-1} = x^{-2}$, and $j_* : \pi_1(U \cap V) \to \pi_1(V)$ maps it to $j_*(1) = 1$ by triviality of $\pi_1(V)$. So by van Kampen,

$$\pi_1(X) \cong \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V) \cong \frac{\langle x \rangle}{\langle x^{-2} \rangle} = \big\langle x \mid x^{-2} = 1 \big\rangle = \big\langle x \mid x^2 = 1 \big\rangle \cong \mathbb{Z}_2.$$

□

Problem 3.

- Letting $\mathsf{Bij}(p^{-1}(x_0))$ denote the set of bijections $p^{-1}(x_0) \to p^{-1}(x_0)$, we have an assignment

  $$F : \pi_1(X, x_0) \to \mathsf{Bij}(p^{-1}(x_0)), \quad F_{[\gamma]}(\tilde{x}) := \tilde{\gamma}_{\tilde{x}}(1),$$

  where $\tilde{\gamma}_{\tilde{x}} : [0,1] \to \tilde{X}$ is the unique lift of γ satisfying $\tilde{\gamma}_{\tilde{x}}(0) = \tilde{x}$. This assignment is precisely the monodromy action of $\pi_1(X, x_0)$ on $p^{-1}(x_0)$, and as such, for any $[\gamma] \in \pi_1(X, x_0)$, the order of $F_{[\gamma]}$ must divide $|\pi_1(X, x_0)| = |\mathbb{Z}_5| = 5$.

- Suppose the cover p is nontrivial. Then there's some $[\gamma] \in \pi_1(X, x_0)$ such that the order of $F_{[\gamma]}$ is not 1. Then by the above, this order must be 5. As such, $F_{[\gamma]}$ is a permutation of 5 distinct elements of $p^{-1}(x_0)$ belonging to a single connected component of $\tilde{X}$. But $\big|p^{-1}(x_0)\big| = 4$, so this is impossible.

□

Problem 4.

Background. A *contact manifold* is a pair (M^{2m+1}, ξ) consisting of an odd-dimensional manifold M together with a "maximally nonintegrable" field of hyperplanes $\{\xi_x \subset \mathsf{T}_x M\}_{x \in M}$, that is, a rank-$2m$ distribution ξ on M which is the kernel of some 1-form $\alpha \in \Omega^1(M)$, called a *contact form*, satisfying $\alpha \wedge (\mathsf{d}\alpha)^{\wedge m} \neq 0$ at each point of M. In this problem we show that $(\mathbb{R}^3, \mathscr{D})$ is a contact manifold.

No. It's enough to show that $\mathscr{D}$ is nonintegrable at $0 \in \mathbb{R}^3$. Let $\alpha := 2\mathsf{d}x - e^y \mathsf{d}z$, so that $\mathscr{D} = \mathsf{ker}(\alpha)$. It's a basic fact from contact geometry that $\mathscr{D}$ is nowhere integrable if $\alpha \wedge (\mathsf{d}\alpha) \neq 0$ at every point of $\mathbb{R}^3$. Indeed,

$$\alpha \wedge (\mathsf{d}\alpha) = (2\mathsf{d}x - e^y \mathsf{d}z) \wedge (-e^y \mathsf{d}y \wedge \mathsf{d}z) = -2e^y \mathsf{d}x \wedge \mathsf{d}y \wedge \mathsf{d}z$$

is nonzero at every point of $\mathbb{R}^3$, and in particular at 0. □

Problem 5.

Suppose M is a submanifold of $\mathbb{R}^4$, and observe that $M = f^{-1}(0)$ where $f : \mathbb{R}^4 \to \mathbb{R}$ is the map given by $f(x_1, x_2, x_3, x_4) := x_1^2 + x_2^2 - x_3^2 - x_4^2$. Consider the tangent spaces of M at two of its points, 0 and $(1, 0, 1, 0)$,

$$\begin{aligned}
&\mathsf{T}_0 M = \mathsf{ker}(\mathsf{d}f_0) = \mathsf{ker}\begin{pmatrix} 2x_1 & 2x_2 & 2x_3 & 2x_4 \end{pmatrix}\Big|_0 = \mathsf{ker}(0) = \mathsf{T}_0\mathbb{R}^4, \\
&\mathsf{T}_{(1,0,1,0)} M = \mathsf{ker}(\mathsf{d}f_{(1,0,1,0)}) = \mathsf{ker}\begin{pmatrix} 2x_1 & 2x_2 & 2x_3 & 2x_4 \end{pmatrix}\Big|_{(1,0,1,0)} = \mathsf{ker}\begin{pmatrix} 2 & 0 & -2 & 0 \end{pmatrix} \\
&\quad = \{(v_1, v_2, v_3, v_4) \in \mathsf{T}_{(1,0,1,0)}\mathbb{R}^4 \mid 2v_1 - 2v_3 = 0\}.
\end{aligned}$$

Then $\mathsf{dim}_{\mathbb{R}}(\mathsf{T}_0 M) = 4$ but $\mathsf{dim}_{\mathbb{R}}(\mathsf{T}_{(1,0,1,0)} M) = 3$, which is impossible. □

Problem 6.

Let U and V be the cylinders along the z- and y-axes, respectively. Then $U \cap V \cong \mathsf{S}^1 \coprod \mathsf{S}^1$, so we have

$$\mathsf{H}_j(U) \cong \mathsf{H}_j(V) \cong \begin{cases} \mathbb{Z} & j = 0, 1, \\ 0 & \text{else}, \end{cases}, \quad \mathsf{H}_j(U \cap V) \cong \begin{cases} \mathbb{Z}^{\oplus 2} & j = 0, 1, \\ 0 & \text{else}. \end{cases}$$

By path connectedness, we already have $\mathsf{H}_0(X) \cong \mathbb{Z}$. Then by Mayer-Vietoris, the sequence

$$0 \longrightarrow \mathsf{H}_2(X) \xrightarrow{\partial_2} \mathbb{Z}^{\oplus 2} \xrightarrow{(i_1, j_1)} \mathbb{Z}^{\oplus 2} \xrightarrow{k_1 - \ell_1} \mathsf{H}_1(X) \xrightarrow{\partial_1} \mathbb{Z}^{\oplus 2} \xrightarrow{(i_0, j_0)} \mathbb{Z}^{\oplus 2}$$

is exact.

- By exactness, $\mathsf{ker}(\partial_2) \cong 0$. Now consider the inclusions $i : U \cap V \hookrightarrow U$ and $j : U \cap V \hookrightarrow V$. The two loops x, y generating $\mathsf{H}_1(U \cap V)$ are mapped under i into contractible portions of the wall of the cylinder U, and so $i_1(x) = i_1(y) = 0$. On the other hand, j sends these two loops to the (same) single loop which generates $\mathsf{H}_1(V)$, and so $j_1(x) = j_1(y) = 1$. Thus $\mathsf{im}(\partial_2) \cong \mathsf{ker}(i_1, j_1) \cong \mathbb{Z}$, and so $\mathsf{H}_2(X) \cong \mathbb{Z}$.

- By the above, $\mathsf{ker}(k_1 - \ell_1) \cong \mathsf{im}(i_1, j_1) \cong \mathbb{Z}$, and so $\mathsf{ker}(\partial_1) \cong \mathsf{im}(k_1 - \ell_1) \cong \mathbb{Z}$. Next observe that the two connected components which together generate $\mathsf{H}_0(U \cap V)$ are mapped by i to the (same) single connected component of U which generates $\mathsf{H}_0(U)$. A similar statement holds for j, whereby $\mathsf{im}(i_0, j_0) \cong \mathbb{Z}$ and $\mathsf{im}(\partial_1) \cong \mathsf{ker}(i_0, j_0) \cong \mathbb{Z}$. Thus $\mathsf{H}_1(X) \cong \mathbb{Z}^{\oplus 2}$.

Hence $\mathsf{H}_j(X) \cong \begin{cases} \mathbb{Z} & j = 0, \\ \mathbb{Z}^{\oplus 2} & j = 1, \\ \mathbb{Z} & j = 2, \\ 0 & \text{else.} \end{cases}$ $\square$

Problem 7.

This is very similar to problem 5 of 2010, Fall, and problem 5 of 2008, Spring.