USC Graduate Exams Complex Analysis

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1 Spring 1992

1. Compute the following integrals

   (a) $\int_{-\infty}^{\infty} \frac{x^2-x+2}{x^4+10x^2+9}dx$

   Proof. Let us integrate this in the complex plane with respect to complex variable z, consider the path γ from $-R$ to R, then moving along Re^{it} for $t \in [0,\pi]$ back to $-R$. Let us consider the contribution of the integral around the path $z = Re^{it}$ for $t \in [0,\pi]$. Changing the variable of integration from z to t yields:

$$\left| \int_{t=0}^{2\pi} \frac{R^2e^{2it} - Re^{it} + 2}{R^4e^{4it} + 10R^2e^{2it} + 9} Rie^{it}dt \right| \leq \int_{t=0}^{2\pi} \left| \frac{R^2e^{2it} - Re^{it} + 2}{R^4e^{4it} + 10R^2e^{2it} + 9} Rie^{it} \right| dt \tag{1.1}$$

$$\leq \int_{t=0}^{2\pi} \frac{M}{R^2} Rdt \tag{1.2}$$

$$= \frac{2\pi M}{R} \tag{1.3}$$

   For some constant $M \in \mathbb{R}$ and all R. Thus as $R \to \infty$, it follows that the integral of $f(z) = \frac{z^2-z+2}{z^4+10z^2+9}$ around $z = Re^{it}$, $t \in [0,\pi]$ goes to 0 as $R \to \infty$. Thus, $\int_{-\infty}^{\infty} \frac{x^2-x+2}{x^4+10x^2+9}dx = \int_\gamma \frac{z^2-z+2}{z^4+10z^2+9}dz$. We can caluclate the integral on the RHS using the calcus of residues noting that $z^4 + 10z^2 + 9 = (z^2+9)(z^2+1) = (z-3i)(z+3i)(z-i)(z+i)$, thus poles that occur on the interior of γ are at $z = i, 3i$, both first order. We now have

$$Res(f, z = i) = \lim_{z \to i}(z-i)f(z) = \frac{i^2 - i + 2}{(i^2+9)(i+i)} = \frac{1-i}{16i} = \frac{-1-i}{16} \tag{1.4}$$

   On the other hand, we also have

$$Res(f, z = 3i) = \lim_{z \to 3i}(z-3i)f(z) = \frac{(3i)^2 - 3i + 2}{(3i+3i)((3i)^2+1)} = \frac{3-7i}{48} \tag{1.5}$$

   By calculus of residues our integral is thus $2\pi i \left(\frac{-1-i}{16} + \frac{3-7i}{48}\right) = \frac{5\pi}{12}$. □

   (b) $\int_{-\pi}^{\pi} \frac{d\theta}{5+4\sin(\theta)}$

   Proof. We let $\theta = 2\tan^{-1}(z)$. Then $d\theta = \frac{2}{1+z^2}dz$. We also have

$$\sin(\theta) = \sin(2\tan^{-1}(z)) = \cos(\tan^{-1}(z))\sin(\tan^{-1}(z)) = \frac{z}{z^2+1} \tag{1.6}$$

   Thus, our integral is equal to

$$2\int_{-\infty}^{\infty} \frac{1}{5+4\frac{z}{z^2+1}} \frac{2}{1+z^2}dz = 4\int_{-\infty}^{\infty} \frac{1}{5+4z+5z^2}dz \tag{1.7}$$

   Let us integrate this in the complex plane with respect to complex variable z, consider the path γ from $-R$ to R, then moving along Re^{it} for $t \in [0,\pi]$ back to $-R$. Let us consider the contribution of the integral around the path $z = Re^{it}$ for $t \in [0, \pi]$. Changing the variable of integration from z to t yields:

$$\left|4\int_{-\infty}^{\infty} \frac{1}{5+4z+5z^2} dz\right| = \left|4\int_{0}^{2\pi} \frac{1}{5+4Re^{it}+5R^2e^{2it}} Re^{it} dt\right| \tag{1.8}$$

$$\leq M \int_0^{2\pi} \frac{1}{R} \tag{1.9}$$

which goes to 0 as $R \to \infty$. Thus the integral around γ as $R \to \infty$ goes towards the integral along the real line from $-\infty$ to ∞. We now use the calculus of residues to evaluate the integral around γ. There is 1 pole located at $z = \frac{-4+\sqrt{4^2-4(5^2)}}{2(5)} = z = \frac{-2+i\sqrt{21}}{5}$.

$$Res\left(z = \frac{-2+i\sqrt{21}}{5}; f(z)\right) = \lim_{z \to \frac{-2+i\sqrt{21}}{5}} \frac{4/5}{z - \frac{-2-i\sqrt{21}}{5}} = \frac{-2i\sqrt{21}}{21} \tag{1.10}$$

And thus the integral is $2\pi i \left(\frac{-2i\sqrt{21}}{21}\right) = \frac{4\pi}{\sqrt{21}}$ $\square$

2. Map the region inside the circle $|z| = 1$ and outside the circle $|z - \frac{1}{2}| = \frac{1}{2}$ conformally onto the unit disk $\{z : |z| < 1\}$.

Proof. Let us first consider the map $f(z) = \frac{1}{z-1}$. Let us consider how this affects the two circles. Every point on $|z| = 1$ can be expressed as a $e^{i\theta}$ for some $\theta \in [0, 2\pi]$. We thus have

$$f(e^{i\theta}) = \frac{1}{e^{i\theta}-1} \tag{1.11}$$

$$= \frac{\cos(\theta) - 1 + i\sin(\theta)}{(\cos(\theta)-1)^2 + \sin^2(\theta)} \tag{1.12}$$

$$= \frac{\cos(\theta) - 1 + i\sin(\theta)}{2 - 2\cos(\theta)} \tag{1.13}$$

$$= -\frac{1}{2} + i\frac{\sin(\theta)}{2-2\cos(\theta)} \tag{1.14}$$

And it is clear that this region is mapped to the vertical line $\text{Re}z = -\frac{1}{2}$. On the other hand, every point on $|z - \frac{1}{2}| = \frac{1}{2}$ is $\frac{1}{2} + \frac{1}{2}e^{i\theta}$ for some $\theta \in [0, 2\pi]$ so that

$$f\left(\frac{1}{2} + \frac{1}{2}e^{i\theta}\right) = \frac{1}{\frac{1}{2} + \frac{1}{2}e^{i\theta} - 1} \tag{1.15}$$

$$= \frac{\cos(\theta) - 1 + i\sin(\theta)}{(\frac{1}{2}\cos(\theta) - \frac{1}{2})^2 + \frac{1}{4}\sin^2(\theta)} \tag{1.16}$$

$$= \frac{\cos(\theta) - 1 + i\sin(\theta)}{\frac{1}{2} - \frac{1}{2}\cos(\theta)} \tag{1.17}$$

$$= -2 + i\frac{\sin(\theta)}{\frac{1}{2} - \frac{1}{2}\cos(\theta)} \tag{1.18}$$

Thus, $f(z)$ maps the given region to a vertical strip between Re$z = -2$ and Re$z = -\frac{1}{2}$. We now just have to map this strip conformally onto the unit disk by $g(z) = \frac{4i}{3}(z + \frac{5}{4})$ which takes this vertical strip to the horizontal strip between Im$z = -1$ and Im$z = 1$. The map $h(z) = e^{\frac{\pi z}{2}}$ will give the right halfplane, and finally, the Mobius transformation $z \mapsto \frac{z-1}{z+1}$ puts the half plane onto the unit disk. Thus, our total transformation is:

$$\frac{h(g(f(z))) - 1}{h(g(f(z))) + 1} = \frac{e^{\frac{\pi g(f(z))}{2}} - 1}{e^{\frac{\pi g(f(z))}{2}} + 1} \tag{1.19}$$

$$= \frac{e^{\frac{\pi \frac{4i}{3}(f(z)+\frac{5}{4})}{2}} - 1}{e^{\frac{\pi \frac{4i}{3}(f(z)+\frac{5}{4})}{2}} + 1} \tag{1.20}$$

$$= \frac{e^{\frac{\pi \frac{4i}{3}(\frac{1}{z-1}+\frac{5}{4})}{2}} - 1}{e^{\frac{\pi \frac{4i}{3}(\frac{1}{z-1}+\frac{5}{4})}{2}} + 1} \tag{1.21}$$

□

3. Determine all entire $f(z)$ and $\Re f(z) > 1$ and $\Im f(z) < -1$, where $\Re$ and $\Im$ denote the real and imaginary part.

Proof. We note that $|e^{-f(z)}| = e^{-\Re} < e^{-1}$ so that $e^{-f(z)}$ is bounded. Since $f(z)$ is entire, it follows that $e^{-f(z)}$ is also entire, and thus by Liouville's Theorem, it follows that $e^{-f(z)}$ is a constant. Thus $f(z)$ must also be a constant. Our final answer is $f(z) = c$ where $c = a + bi$ for some $a > 1$ and $b < -1$. □

4. How many roots does the equation

$$z^{15} - 2z^{11} + 7z^3 - 2z^2 + 1 = 0 \tag{1.22}$$

have in the unit disk $|z| < 1$.

Proof. Let $g(z) = z^{15} - 2z^{11} + 7z^3 - 2z^2 + 1$ and consider $f(z) = 7z^3 + 1$. Then when $|z| = 1$, we have $|f(z)| \geq 6$. On the other hand $|f(z) - g(z)| = |z^{15} - 2z^{11} - 2z^2| \leq 1 + 2 + 2 < |f(z)|$. Thus, by Rouche's theorem, it follows that $f(z)$ and $g(z)$ have the same number of roots. It' clear that $f(z) = 0$ implies $z = -\frac{1}{\sqrt[3]{7}}, -\frac{1}{\sqrt[3]{7}}e^{2\pi/3}, -\frac{1}{\sqrt[3]{7}}e^{4\pi/3}$, all of which are in the unit disk. The given function thus has 3 roots in the unit disk. □

2 Spring 1993

1. Compute the following integral using residues

$$\int_0^\infty \frac{x^{\frac{1}{2}} \log x}{(x^2 + 1)^2} dx \tag{2.1}$$

Proof. □

2. Map the upper half disk $\{z : |z| < 1, \Im z > 0\}$ conformally onto the unit disk $\{z : |z| < 1\}$.

   Proof. Let us consider the mapping $f(z) = z + \frac{1}{z}$ which maps the given region to the upper half plane. Now $g(z) = iz$ maps the upper half plane to the right half plane. Finally $h(z) = \frac{z-1}{z+1}$ maps the right half plane to the unit circle. Thus the composition yields:

$$h(g(f(z))) = \frac{g(f(z)) - 1}{g(f(z)) + 1} \tag{2.2}$$
$$= \frac{if(z) - 1}{if(z) + 1} \tag{2.3}$$
$$= \frac{iz + \frac{i}{z} - 1}{iz + \frac{i}{z} + 1} \tag{2.4}$$
$$= \frac{iz^2 + i - z}{iz^2 + i + z} \tag{2.5}$$

□

3. How many roots does the equation

$$z^{11} + 4z^{10} - z^9 + 12z^5 - 2z^4 + z - 1 = 0 \tag{2.6}$$

   have in the annulus $1 < |z| < 2$.

   Proof. We count the number of roots inside $|z| < 2$ and subtract it from the number of roots inside $|z| < 1$.

   First let $f(z) = z^{11} + 4z^{10} \geq 2^{11}$ and $g(z)$ the given function. Then $|f(z) - g(z)| = | - z^9 + 12z^5 - 2z^4 + z - 1| \leq 2^9 + 12 \cdot 2^5 + 2^4 + 2 + 1 = 2^9 + 2^8 + 2^7 + 2^4 + 2 + 1 < |f(z)|$. Thus, we can apply Rouche's theorem and conclude that $g(z)$ has 10 roots in $|z| < 2$.

   We now subtract from the roots that are in $|z| < 1$. Let $f(z) = 4z^{10} + 12z^4$ so that $|f(z) - g(z)| = |z^{11} - z^9 - 2z^4 + z - 1| \leq 6 \leq 12 - 4 \leq |f(z)|$. Thus by Rouche's theorem, $g(z)$ must have 4 roots in the unit disk. Combining everything there must be $10 - 4 = 6$ in the annulus. □

3 Fall 1993

1. Define $D_r = \{z \in \mathbb{C} : |z| < r\}$, the open r-disk. Let $M > 0$ and $f_n : D_1 \to D_M$ for $n = 1, 2, ...$ be a sequence of analytic functions. Prove there is a subsequence which converges uniformly on $D_{1/2}$.

   Proof. Since f_n analytic on the complex space, it is also holomorphic. It's also clear that D_r is uniformly bounded Since $D_{1/2}$ is compact, can apply Montel's therem if we can prove that f_n is □

2. Prove or find a counterexample: Let D be a coutable dense subset of $(0, 1)$ and let G be an open subset of $\mathbb{R}$ such that $G \supset D$, then $G \supset (0, 1)$.

   Proof. We shall find a counterexample. Let D be the rational numbers in $(0, 1)$, which is dense and countable in $(0, 1)$. Now let $G = (0, \frac{\sqrt{2}}{2}) \cup (\frac{\sqrt{2}}{2}, 1)$. It's clear that $G \supset D$, and that G is open. However, G doesn't contain $(0, 1)$, thus giving us our counterexample. □

3. Let f be a non-constant meromorphic function with is doubly periodic (i.e. has two periods linearly independent over the reals). Prove that f has at least one singularity.

   Proof. Since the reals is one dimensional vector space, I assume they mean "has two periods linearly independent over the complex numbers". Let the periods be $a, b \in \mathbb{C}$. Then every point in the complex plane can be expressed as $z = xa + yb$ for some $x, y \in \mathbb{R}$. Since f is periodic, we have $f(xa + yb) = f((x \pm 1)a, y(\pm 1)b)$. Thus, we have $f(z) = f(z')$ for some z' within the parallelogram with vertices $0, a, a+b, b$. If there is no singularity, since this is a compact set f will reach its maximum on this parallelogram, and thus f will have a global maximum. However, by the maximum principles, f, being a complex valued function, cannot attain a maximum on an open set. Thus f has at least one singularity. □

4. How many roots of the equation $f(z) = 0$ lie in the right half-plane, where

$$f(z) = z^4 + \sqrt{2}z^3 + 2z^2 - 5z + 2 \tag{3.1}$$

   Proof. We let our contour be the curve from iR to $-iR$, and around the semicircle of radius R in the right half plane. If we set $g(z) = z^4 + 2$ then $|g(z)| \geq R^4 - 2$ when $|z| = R$ On the other hand, when z is on the imaginary axis, we have:

$$f(it) = t^4 - i\sqrt{2}t^3 - 2t^2 - 5it + 2 \tag{3.2}$$

   for $t \in \mathbb{R}$. We let our o$|f(it)| = \sqrt{(t^4 - 2t^2 + 2)^2 + (\sqrt{2}t^3 + 5t)^2}$. Let $g(z) = z^4 - 5z$ Let $g(z) = z^4 + 2$ □

5. Show that a function $f : (a, b) \to \mathbb{R}$ which is absolutely continuous is both uniformly continuous and of bounded variation.

   Proof. Since f is absolutely continuous, for all $\epsilon > 0$, there exists $\delta > 0$ such that if a sequence of pairwise disjoint subintervals (x_k, y_k) of (a, b) satisfy $\sum_k (y_k - x_k) < \delta$, then $\sum_k |f(y_k) - f(x_k)| < \epsilon$. Letting $k = 1$ proves that f is uniformly continuous.

   Now we prove that f has bounded variation. Let define $var_{(a,b)}f$ to be the variation of f on the interval (a, b). By hypothesis, there is a $\delta > 0$ such that $\sum_k |f(y_k) - f(x_k)| < 1$ for all disjoint subintervals

(x_i, y_i) and $\sum_k |y_k - x_k| < \delta$. Let N be an integer greater than $\frac{b-a}{\delta}$, and partition (a,b) into N evenly space intervals $\left(a + \frac{j(b-a)}{N}, a + \frac{(j+1)(b-a)}{N}\right)$. Thus, $var_{(a,b)} f = \sum_{j=1}^{N} var_{\left(a+\frac{j(b-a)}{N}, a+\frac{(j+1)(b-a)}{N}\right)} < \sum_{j=1}^{N} 1 = N$. □

6. Show that $\frac{\sin x}{x} \in L^2(\mathbb{R}^+)$ and evaluate its L^2 norm.

Proof. We first show $\left(\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx\right)^{\frac{1}{2}} < \infty$. By Lhopital's rule, $\lim_{x\to 0} \frac{\sin x}{x} = 1$ so by continuity of $\frac{\sin x}{x}$, it follows that $\left(\int_0^1 \left(\frac{\sin x}{x}\right)^2 dx\right)^{\frac{1}{2}} < \infty$. Now we just have to show that $\left(\int_1^\infty \left(\frac{\sin x}{x}\right)^2 dx\right)^2 < \infty$. This follows from

$$\left(\int_1^\infty \left(\frac{\sin x}{x}\right)^2 dx\right)^{\frac{1}{2}} < \left(\int_1^\infty \frac{1}{x^2} dx\right)^{\frac{1}{2}} < \infty = 1 < \infty \tag{3.3}$$

Now to evaluate this integral, we can use calculus of residues. Since this function is even we can Evaluated it from $-\infty$ to ∞ and divide the result by 2. □

7. Suppose f is a non-negative function which is Lebesgue integrable on $[0,1]$, and $\{r_n : n = 1, 2, ...\}$ is an enumeration of the rational numbers in $[0,1]$. Show that the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} f(|x - r_n|) \tag{3.4}$$

converges for a.e. $x \in [0,1]$.

Proof. We will set $g(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} f(|x - r_n|)$, and prove that $\int_0^1 g(x)dx < \infty$. From this it will immediately follow that $g(x) < \infty$ a.e. First, we note that $\int_0^1 f(|x - r_n|) \leq \int_{-1}^1 f(|x - r_n|) = 2\int_0^1 f(x)$. Thus:

$$\int_0^1 g(x)dx = \sum_{n=1}^{\infty} \int_0^1 \frac{1}{2^n} f(|x - r_n|) \tag{3.5}$$

$$\leq 2\int_0^1 f(x)dx \sum_{n=1}^{\infty} \frac{1}{2^n} \tag{3.6}$$

$$= 2\int_0^1 f(x) < \infty \tag{3.7}$$

as desired. □

4 Spring 1994

1. Evaluate $\int_0^\infty \frac{\log x}{1+x^2} dx$.

Proof. Let us integrate this in the complex plane with respect to complex variable z, consider the path γ from $-R$ to R, then moving along Re^{it} for $t \in [0, \pi]$ back to $-R$. Let us consider the contribution of the integral around the path $z = Re^{it}$ for $t \in [0, \pi]$. Changing the variable of integration from z to t yields:

$$\left| \int_0^\infty \frac{\log x}{1+x^2} dx \right| < M \int_0^{2\pi} \frac{\log R}{|1+R^2|} dt \tag{4.1}$$

$$\leq M \frac{\log R}{R} \tag{4.2}$$

for some constant M and sufficiently large R. Thus, the integral around γ reduces to the portion of the integral on the real number line as $R \to \infty$. We can use the calculus of residues to evaluate this integral. There is only 1 reside at $z = i$ evaluated as follows:

$$Res(f(z); z = i) = \lim_{z \to i} (z - i) \frac{\log z}{(z-i)(z+i)} \tag{4.3}$$

$$= \tag{4.4}$$

□

2. Show that $[0, 1]$ cannot be written as the countably infinite union of disjoint nonempty closed intervals.

Proof. We prove this statement by contradiction. Suppose $[0, 1]$ was the union of countably many closed intervals. Then removing the endpoints of each interval we get that there is a sequence of disjoint open intervals I_n such that

$$[0, 1] = \cup_{n=1}^\infty I_n \tag{4.5}$$

Letting $I_n = [x_n, y_n]$ we consider the union of the endpoints;

$$U = \cup_{n=1}^\infty \{x_n, b_n\} \tag{4.6}$$

U is clearly closed, and we can see that U is also perfect since every point in U is a limit point. We can now apply the Baire category theorem which shows that a perfect subset of a complete metric space can't be countable infinite. The result follows. □

3. Let $f : D \to \mathbb{C}$ be analytic such that $\Re f(z) > 0$ for all z. Prove

$$|f(z)| \leq |f(0)| \frac{1+|z|}{1-|z|} \tag{4.7}$$

Proof. We note the map $g(z) = \frac{f(0)-z}{f(0)+z}$ maps the right half complex plane conformally onto the unit disk such that $g(f(0)) = 0$. Thus, we can apply Schwarz's lemma to the function $g(f(z))$ to obtain $|g(f(z))| \leq |z|$. We also have

$$g^{-1}(z) = f(0) \frac{1-z}{1+z} \tag{4.8}$$

Thus, $|f(z)| \leq g^{-1}(|z|) = f(0)\frac{1-|z|}{1+|z|}$ as desired. □

4. Let $f : [1, +\infty) \to [0, +\infty)$ be Lebesgue measurable. Prove:

$$\int_1^\infty \frac{f(x)^2}{x^2} < +\infty \Rightarrow \int_1^\infty \frac{f(x)}{x^2} dx < +\infty \tag{4.9}$$

Proof. Define $S_0 = \{x : f(x) < 1\}$, and $S_1 = \{x : f(x) \geq 1\}$. It's clear that $\int_1^\infty \frac{f(x)}{x^2} dx = \int_{S_0} \frac{f(x)}{x^2} dx + \int_{S_1} \frac{f(x)}{x^2}$, so if we can bound each of the integrals then we are done.

First, we have $\int_{S_0} \frac{f(x)}{x^2} dx \leq \int_{S_0} \frac{1}{x^2} dx < \int_1^\infty \frac{1}{x^2} dx < \infty$.

On the other hand, we have $\int_{S_1} \frac{f(x)}{x^2} < \int_{S_1} \frac{f(x)^2}{x^2} < \int_1^\infty \frac{f(x)^2}{x^2} < \infty$ by hypothesis. Putting everything together yields our desired result. □

5.

6. Let $([0,1], \mathcal{A}, \mu)$ denote the Lebesgue space on $f : [0,1] \to \mathbb{R}$ the condition "f is continuous a.e." neither implies, nor is implied by, the condition "there exists a continuous function $g : [0,1] \to \mathbb{R}$ such that $f = g$ a.e."

Proof. Let $g(x) = 0$ and define $f(x)$ as follows:

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases} \tag{4.10}$$

Then because $\mathbb{Q}$ has Lebesgue measure 0, it follows that $g(x) = f(x)$ a.e. However, $f(x)$ is nowhere continuous.

Conversely, let

$$f(x) = \begin{cases} 0 & \text{if } x \leq \frac{1}{2} \\ 1 & \text{if } x > \frac{1}{2} \end{cases} \tag{4.11}$$

which will always differ from a continuous function g around an interval centered at $x = \frac{1}{2}$, and thus not equal to g a.e. □

7. An entire function is said to have finite order if there exists $c > 0$ such that $|f(z)| \leq exp(|z|^c)$ for all $|z|$ sufficiently large; the order of f is the infimum of all such $c > 0$. Prove that the following function is entire and has order $\frac{1}{2}$.

$$f(z) = \prod_{k=1}^{\infty} \left(1 + \frac{z}{k^2}\right) \tag{4.12}$$

Proof. □

8. Let $\{f_n\}$ be a sequence of measurable functions on some measure space $(X, \mathcal{A}, \mu)$ with $\mu(X) < \infty$. We say the sequence is uniformly integrable if

$$\lim_{n\to\infty} \sum_n \int_{|f_n|>R} |f_n| d\mu = 0 \tag{4.13}$$

(a) Show that if there exists $g \in L^1(X)$ such that $|f_n(x)| \leq |g(x)|$ for all x, n then the $\{f_n\}$ are uniformly integrable.

Proof. Since $\mu(X) < \infty$, and $g \in L^1(X)$, it follows that $ess\sup_{x\in X} g(x) < \infty$. Thus, whenever $R > ess\sup_{x\in X} g(x)$, $\int_{|f_n|>R} |f_n| d\mu = 0$ for all n, and thus $\lim_{n\to\infty} \sum_n \int_{|f_n|>R} |f_n| d\mu = 0$ as desired. □

(b) Prove that if $f_n \to f$ pointwise and the $\{f_n\}$ are uniformly integrable then $f \in L^1(\mathbb{R})$ and

$$\lim_n \int f_n d\mu = \int f d\mu \tag{4.14}$$

Proof. We have the following inequality:

$$\int_X |f| d\mu = \int_{|f|\leq R} |f| + \int_{|f|>R} |f| d\mu < \int_{|f|\leq R} |f| + \sum_n \int_{|f_n|>R} |f_n| d\mu < \infty \tag{4.15}$$

where in the last inequality follows we assume R is sufficiently large such that $\sum_n \int_{|f_n|>R} |f_n| < \infty$. Thus $f \in L^1(X)$ and by the Lebesgue dominated convergence theorem, it follows that $\lim_n \int f_n d\mu = \int f d\mu$ as desired. □