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Complex Analysis Exam Cheat Sheet

Cauchy Formulas

Theorem 1. *Cauchy-Riemann Equations*

$f(z) = f(x, y) = u(x, y) + iv(x, y)$ is analytic (C^∞) in Ω $\boxed{\iff}$ $u_x = v_y$, $u_y = -v_x$

And $f'(x) = u_x + iv_x$.

Theorem 2. *Cauchy Integral Formula*

$\boxed{\text{If:}}$ $f(z)$ is analytic on open simply connected region Ω containing $\{|\xi - z| < \rho\}$,

$\boxed{\text{Then:}}$

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_{|\xi - z|<\gamma} \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi$$

***Note that if f is analytic in Ω, then $f(z) = \dfrac{1}{2\pi i}\int$

Theorem 3. *Cauchy Estimate*

$\boxed{\text{If:}}$ $f(z)$ is analytic in Ω containing $B = \{|\xi - z| < R\}$,

$\boxed{\text{Then:}}$ if $M_R = \max\limits_{a\in\partial B} |f(a)|$ then

$$|f^{(n)}(z)| \leq \frac{n!M_R}{R^n}$$

Counting Zeros

Theorem 4. *Rouche's*

If:

- f and g are analytic in Ω containing a closed curve Γ
- $|f(z) - g(z)| < |f(z)|$ or $|f(z) - g(z)| < |g(z)|$ for all $z \in \Gamma$,

Then: f and g have the same number of zeros inside Γ.

***Alternatively, if $|g(z)| < |f(z)|$ on Γ, then f and $f + g$ have the same number of zeros inside Γ.

Theorem 5. *Argument Principle*

If: f is meromorphic in Ω and $\Gamma \subset \Omega$ has winding number 1 and avoids zeros and poles of f,

Then: then

$$\frac{1}{2\pi i}\int_\Gamma \frac{f'(z)}{f(z)}dz = \#\text{zeros} - \#\text{poles} \qquad \text{enclosed by } \Gamma.$$

Example 1.

Find the number of solutions of the equation $z - 2 - e^{-z} = 0$ in $H = \{z \in \mathbb{C} : \mathfrak{Re}(z) > 0\}$.

Let $f(z) = z - 2 - e^{-z}$. Then $f(iy) = iy - 2 - e^{-iy} = -2 - \cos(y) + i(y - \sin(y))$. Thus, $\mathfrak{Re}(f(iy)) < 0$ for all $y \in \mathbb{R}$ and f sends the imaginary axis to the left-half plane, away from the origin.

On a large half-circle in the right-half plane, $z = Re^{i\theta}$ for $\theta \in (-\pi/2, \pi/2)$ and

$$\frac{1}{R}f(Re^{i\theta}) = e^{i\theta} - \frac{2}{R} - \frac{e^{-Re^{i\theta}}}{R} \to e^{i\theta} \qquad R \to \infty.$$

Therefore, f has a total change in argument of π so f can have at most one zero in the right half plane.

Since $f(0) < 0$ and $f(10) = 8 - \frac{1}{e^{10}} > 7 > 0$ by the intermediate value theorem, f has a zero on the positive real axis so f has exactly one zero in the right-half plane.

Bounded Functions

Theorem 6. *Louiville*

If: $f(z)$ is entire and bounded

Then: $f(z)$ is constant

Theorem 7. *Schwarz'*

If: $f : \mathbb{D} \to \mathbb{D}$ ($\mathbb{D} = \{|z| < 1\}$)

- analytic
- $f(0) = 0$

Then: $|f(z)| \leq |z|$ on $\mathbb{D}$ and $|f'(0)| \leq 1$.

***If, additionally $|f(z)| = |z|$ for some $z \neq 0$ *or* if $|f'(0)| = 1$, then $f(z) = az$ for some $|a| = 1$.

Theorem 8. *Maximum Modulus Principle*

If: If f is analytic in an open simply connected set Ω, and f has a maximum value inside Ω

Then: f is constant.

***Namely, analytic function must attain their maximum on the boundary of any simply connected set.

Theorem 9. *Schwarz Reflection Principle*

If:

- f is analytic in the upper half plane,
- f is continuous on the real line and $f(x) \in \mathbb{R}$ for all $x \in \mathbb{R}$ (f is real on the real line)

Then: f can be extended to an analytic function on the negative half plane by the formula $f(\overline{z}) = \overline{f(z)}$.

Residues

Theorem 10. *Residue Theorem*

If:

- $\Gamma \subset \Omega$ closed curve
- Ω open and simply connected,
- $f(z)$ is a meremorphic function in Ω
- Γ intersects no poles of f

Then: if Γ encloses $\{a_1, ..., a_n\}$ poles of f,

$$\int_\Gamma f(z)dz = 2\pi i \sum_{j=1}^{n} \text{Res}_{z=a_j} f(z)$$

Formula 1. *Residue Formula* If a is a pole of order n of $f(z)$, then

$$\text{Res}_{z=a} f(z) = \lim_{z\to a} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}}((z-a)^n f(z)).$$

***The residue of $f(z)$ at a is exactly the coefficient of the $\frac{1}{z}$ term in the Laurent expansion of $f(z)$ at a.

Example 2.

Evaluate
$$\int_0^{2\pi} \frac{d\theta}{3+\cos\theta+2\sin\theta}.$$

$$\begin{aligned}
\int_0^{2\pi} \frac{d\theta}{3+\cos\theta+2\sin\theta} &= \int_0^{2\pi} \frac{d\theta}{3+\frac{e^{i\theta}+e^{-i\theta}}{2}+\frac{e^{i\theta}-e^{-i\theta}}{i}} \\
&= \int_0^{2\pi} \frac{2ie^{i\theta}d\theta}{6ie^{i\theta}+ie^{2i\theta}+i+2e^{2i\theta}-2} \\
&= \int_{|z|=1} \frac{2dz}{6iz+iz^2+i+2z^2-2} \qquad z=e^{i\theta} \\
&= \int_{|z|=1} \frac{2dz}{(i+2)z^2+6iz+i-2} \\
&= \int_{|z|=1} \frac{2dz}{(i+2)\left(z+\frac{1}{5}(1+2i)\right)(z+1+2i)} \qquad (1) \\
&= \left(2\pi i \text{Res}_{z=-\frac{1}{5}(1+2i)} \frac{2}{(i+2)\left(z+\frac{1}{5}(1+2i)\right)(z+1+2i)}\right) \qquad (2) \\
&= 2\pi i \frac{2}{(i+2)\left(-\frac{1}{5}(1+2i)+1+2i\right)} \\
&= 4\pi i \frac{1}{(i+2)\frac{4}{5}(1+2i)} \\
&= 5\pi i \frac{1}{(1+2i)(i+2)} \\
&= 5\pi i \frac{1}{i+2-2+4i} \\
&= 5\pi i \frac{1}{5i} \\
&= \pi
\end{aligned}$$

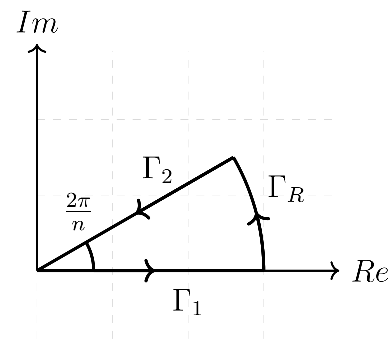
Where (1) comes from the quadratic formula, and (2) because only one pole is contained in the unit disk.

Example 3.

Evaluate the integral

$$\int_0^\infty \frac{dx}{1+x^n}, \qquad n \geq 2.$$

Then we get that there is a pole at $e^{i\frac{\pi}{n}}$ which can be isolated in a pizza slice of angle $\frac{2\pi}{n}$. Thus, we integrate around the following contour:



Immeidately, we get that

$$|I_R| = \left|\int_{\Gamma_R} \frac{dz}{1+z^n}\right| \leq \int_0^{\frac{2\pi}{n}} \frac{R}{R^n - 1} d\theta = \frac{2\pi R}{n(R^n-1)} \to 0 \qquad R \to \infty.$$

Since

$$I_2 = \int_{\Gamma_2} \frac{dz}{1+z^n} = \int_R^0 \frac{e^{i\frac{2\pi}{n}} dr}{1 + r^n e^{2\pi i}} = -e^{i\frac{2\pi}{n}} \int_0^R \frac{dr}{1+r^n} = -e^{i\frac{2\pi}{n}} I_1.$$

Thus, using the residue theorem, we get that

$$\text{Res}_{z=e^{i\frac{\pi}{n}}} \frac{1}{1+x^n} = \lim_{z \to e^{i\frac{\pi}{n}}} \frac{x - e^{i\frac{\pi}{n}}}{x^n + 1} = \lim_{z \to e^{i\frac{\pi}{n}}} \frac{1}{nx^{n-1}} = \frac{1}{n} e^{-(n-1)i\frac{\pi}{n}}$$

and that

$$2\pi i \frac{1}{n} e^{-(n-1)i\frac{\pi}{n}} = \lim_{R\to\infty} (I_1 + I_2 + I_R) = (1 - e^{i\frac{2\pi}{n}}) I_1$$

and so

$$\int_0^\infty \frac{dx}{1+x^n} = \frac{\pi}{n} e^{-(n-1)i\frac{\pi}{n}} \frac{2i}{1 - e^{i\frac{2\pi}{n}}} = \frac{\pi}{n} \frac{-2i}{e^{-i\frac{\pi}{n}} - e^{i\frac{\pi}{n}}} = \frac{\pi/n}{\sin(\pi/n)}$$

Harmonic and Subharmonic

Definition 1. *Harmonic Function*

$u : \Omega \to \mathbb{R}$ where Ω is open and could be real or complex is harmonic if its Laplacian is zero:

$$u_{xx} + u_{yy} = 0.$$

Theorem 11. *Maximum and Minimum Principle*

If: u is harmonic on an open simply connected set Ω and u attains a maximum *or* minimum value inside Ω

Then: u is constant

***harmonic functions attain their maximum *and* minimum values on the boundary of open sets.

Theorem 12. *Mean Value Property*

If: u is harmonic on an open set Ω,

Then: for each $z_0 \in \Omega$ and $r > 0$ such that $\overline{B_r(z_0)} = \{|z - z_0| \leq r\} \subset \Omega$,

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})d\theta.$$

Theorem 13. *Poisson Formula*

If: u is harmonic on an open set Ω and $\overline{\mathbb{D}} = \{|z| \leq 1\} \subset \Omega$,

Then: for all $|z| < 1$,

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} u(e^{i\theta})d\theta.$$

***The Poisson Kernel is $\dfrac{|\xi|^2 - |z|^2}{|\xi - z|^2}$.

Theorem 14. *Harnack's Inequality*

If: u is a *non-negative* analytic inside $B_R(z_0)$ and continuous on the boundary,

Then: for all $r < R$,

$$\frac{R-r}{R+r}u(z_0) \leq u(z) \leq \frac{R+r}{R-r}u(z_0).$$

Definition 2. *Subharmonic Function*

$v : \Omega \to \mathbb{R}$ where Ω is open and could be real or complex is subharmonic if its Laplacian is non-negative:

$$u_{xx} + u_{yy} \geq 0.$$

Theorem 15. *Maximum Principle for Subharmonic Functions*

If: v is subharmonic in Ω, and u is any harmonic function in Ω

Then: $u - v$ has the maximum principle (but not necessarily the minimum principle).

Theorem 16. *MVP for Subharmonic Functions*

If: v is subharmonic on Ω

Then: for each $z_0 \in \Omega$ and $r > 0$ such that $\overline{B_r(z_0)} = \{|z - z_0| \leq r\} \subset \Omega$,

$$v(z_0) \leq \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})d\theta.$$

Infinite Series and Products

Theorem 17. *Taylor's*

$f(z)$ is analytic at a point z_0 $\iff$ There exists a neighborhood U of z_0 such that f has a convergent Taylor Series at z_0 in U.

The Taylor Series is unique and converges uniformly on compact subsets.

Theorem 18. *Laurent*

$f(z)$ has an isolated singularity at z_0 $\iff$ There exists Laurent series at z_0 which converges in some annulus avoiding z_0.

The Laurent Series is unique and converges uniformly on compact subsets.

Theorem 19. *Convergence Criterion for Infinite product*

$\prod_{n=1}^{\infty} a_n$ converges $\iff$ $\sum_{n=1}^{\infty} \log(a_n)$ converges for some branch cut of $\log$.

$\prod_{n=1}^{\infty}(1+a_n)$ converges absolutely $\iff$ $\sum_{n=1}^{\infty}|a_n|$ converges

Theorem 20. *Analyticitiy of an Infinite product*

$\prod_{n=1}^{\infty} f_n(z)$ represents an analytic function $\iff$ $\prod_{n=1}^{\infty} f_n(z)$ converges uniformly on compact subsets.

$\prod_{n=1}^{\infty} f_n(z)$ converges uniformly on compact subsets $\iff$ $\sum_{n=1}^{\infty}|f_n(z)-1|$ converges uniformly on compact subsets

Theorem 21. *Weierstrauss M-test*

If: there exists a sequence $\{M_n\}_{n=1}^{\infty} \subset \mathbb{R}^+$ ($M_n \geq 0$ for all n) such that $|f_n(z)| \leq M_n$ for all n

Then: $\sum_{n=1}^{\infty} f_n(z)$ converges absolutely and uniformly on compact subsets.

Definition 3. *Radius of Convergence*

The radius of convergence R of a series (Taylor or otherwise) $\sum_{n=1}^{\infty} a_n(z-z_0)^n$ is given by the formula $\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n}$

Formula 2. *Taylor Series*

- $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad |z|<1$
- $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$
- $\log(1-z) = \sum_{n=1}^{\infty} \frac{z^n}{n} \qquad |z|<1$
- $\sin(z) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$
- $\cos(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$

Example 4.

Write an entire function which has the simple zeros $1, 4, 9, 16, 25, \ldots$ and has no other zeros.

The zeros are $1^2, 2^2, 3^2, 4^2, 5^2, \ldots$. Then let

$$f(z) = \prod_{n=1}^{\infty} \frac{(z-n^2)}{n^2} = \prod_{n=1}^{\infty}\left(\frac{z}{n^2}-1\right) = -\prod_{n=1}^{\infty}\left(1-\frac{z}{n^2}\right)$$

Since for all z, there exists M so $|z| < M$, we have that $\frac{|z|}{n^2} < \frac{M}{n^2}$ and so

$$\sum_{n=1}^{\infty} \frac{|z|}{n^2}$$

converges uniformly on compact subsets. Therefore, f is entire.

Example 5. Let

$$f(z) = \frac{1}{z(z+1)}.$$

Then f has singularities at -1 and 0. Namely, $f(z)$ will have a Laurent series expansion for $0 < |z| < 1$ and for $|z| > 1$.

On $0 < |z| < 1$,

$$\begin{aligned} f(z) &= \frac{1}{z(z+1)} \\ &= \frac{1}{z} - \frac{1}{z+1} \\ &= \frac{1}{z} - \frac{1}{1-(-z)} \\ &= \frac{1}{z} - \sum_{n=0}^{\infty}(-z)^n \\ &= \sum_{n=-1}^{\infty} (-1)^{n+1} \frac{1}{z^n} \end{aligned}$$

On $|z| > 1$, $\frac{1}{|z|} < 1$, so

$$\begin{aligned} f(z) &= \frac{1}{z(z+1)} \\ &= \frac{1}{z} - \frac{1}{z+1} \\ &= \frac{1}{z} - \frac{1}{z(1+\frac{1}{z})} \\ &= \frac{1}{z} - \frac{1}{z}\frac{1}{1-\left(-\frac{1}{z}\right)} \\ &= \frac{1}{z} - \frac{1}{z}\sum_{n=0}^{\infty}(-1)^n \frac{1}{z^n} \\ &= \frac{1}{z} + \sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{z^{n+1}} \\ &= \sum_{n=2}^{\infty}(-1)^n \frac{1}{z^n} \end{aligned}$$

Now, if we were being asked to find the Laurent Expansion on $\{1 < |z-1| < 2\}$, then we would be being asked to find the expansion at $a = 1$. Since on $\{1 < |z-1| < 2\}$ we have that $1 > \frac{1}{|z-1|} > \frac{1}{2}$ and $\frac{|z-1|}{2} < 1$ so

$$\begin{aligned}
\frac{1}{z(z+1)} &= \frac{1}{z} - \frac{1}{z+1} \\
&= \frac{1}{(z-1)+1} - \frac{1}{(z-1)+2} \\
&= \frac{\frac{1}{z-1}}{1+\frac{1}{z-1}} - \frac{\frac{1}{2}}{1+\frac{z-1}{2}} \\
&= \frac{1}{z-1}\frac{1}{1-\frac{1}{1-z}} - \frac{1}{2}\frac{1}{1-\frac{1-z}{2}} \\
&= \frac{1}{z-1}\sum_{l=0}^{\infty}\left(\frac{1}{1-z}\right)^l - \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{1-z}{2}\right)^k \\
&= \sum_{l=0}^{\infty}\frac{1}{(1-z)^{l+1}} - \sum_{k=0}^{\infty}\frac{(1-z)^k}{2^{k+1}}
\end{aligned}$$

Singularities

Definition 4. *Singularities*

- **Exceptional Point:** $\lim_{z\to a}(z-a)f(z)=0$.
- **Zeros:** $f(a)=0$, there exists k and g analytic (and nonzero at a) such that $f(z)=(z-a)^k g(z)$.
- **Poles:** $|f(a)|=\infty$, there exists a k and g analytic such that $f(z)=\dfrac{g(z)}{(z-a)^k}$.
- **Essential Singularity:** Any isolated singularity that is not a pole and is not removable.

***Exceptional points are exactly removable singularities. Namely, if $f(z)$ has an exceptional point, it can be extended to an analytic function at that point.

***If $f(z)$ has any type of removable singularity at ∞, then $f(1/z)$ has a singularity of the *same type* at 0.

Theorem 22. *Picard's Little Theorem*

If: $f(z)$ is entire and non-constant

Then: f assumes all but at most 1 point in $\mathbb{C}$.

Theorem 23. *Picard's Great Theorem*

If: $f(z)$ has an essential singularity z_0,

Then: for every punctured neighborhood U of z_0, $f(z)$ for $z\in U$ assumes all but at most 2 points in $\mathbb{C}\cup\{\infty\}$ infinitely often.

Normal Families

Definition 5. *Singularities*

If $\mathcal{F}$ is a family (set) of holomorphic functions $f : \Omega \to \mathbb{C}$ is called *normal* if for every sequence $\{f_n\}_{n=1}^\infty \subset \mathcal{F}$ there exists a subsequence $\{f_{n_k}\}_{k=1}^\infty$ which converges uniformly on compact subsets of Ω.

***Note that any sequence of holomorphic functions converging uniformly must converge to a holomorphic function.

Theorem 24. *Montel's*

If $\mathcal{F}$ is a family of holomorphic functions on an open set Ω, then

$\mathcal{F}$ is normal	$\boxed{\iff}$	$\mathcal{F}$ is locally uniformly bounded (for each compact subset K of Ω, there exists M so $\lvert f(z)\rvert \leq M$ for all $z \in K$ and for all $f \in \mathcal{F}$)

Conformal Mapping

Definition 6. *Conformal Map*

A map f on an open region Ω is conformal if

- f is analytic on Ω
- $f'(z) \neq 0$ for all $z \in \Omega$

***Conformal maps are angle preserving.

Theorem 25. *Identity Theorem*

If: $f, g : \Omega \to \mathbb{C}$ where Ω is open and simply connected and $f = g$ on some subset $S \subset \Omega$ having an accumulation point in Ω

Then: $f = g$ on all of Ω.

Theorem 26. *Open Mapping Theorem*

If: f is analytic

Then: the image of any open set under f is also open (f sends open sets to open sets).

Theorem 27. *Riemann Mapping Theorem*

If: $\Omega \subset \mathbb{C}$ is

- open,
- simply connected,
- and not all of $\mathbb{C}$ (Ω lacks at least one point of $\mathbb{C}$ or lacks at least two points of $\overline{\mathbb{C}}$)

Then: For each $\alpha \in \Omega$ there exists a unique conformal biholomorphic (bijective, analytic, with analytic inverse) function

$$g : \Omega \to \mathbb{D} = \{|z| < 1\} \qquad g(\alpha) = 0$$

Definition 7. *Cross Ratio*

The cross ratio

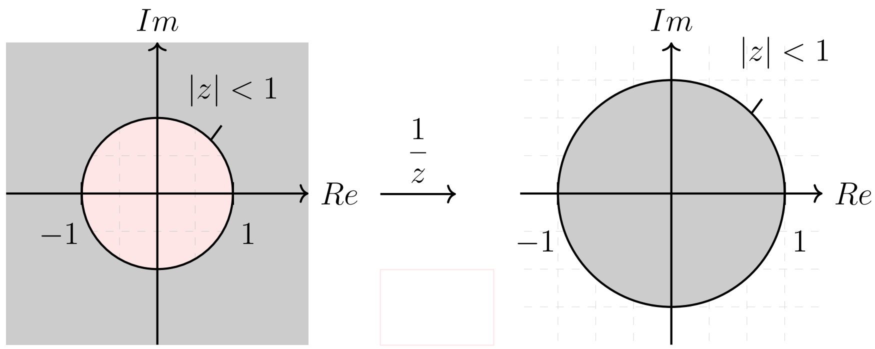
$$(z, w_2, w_3, w_4) = \frac{z - w_3}{z - w_4} : \frac{w_2 - w_3}{w_2 - w_4}.$$

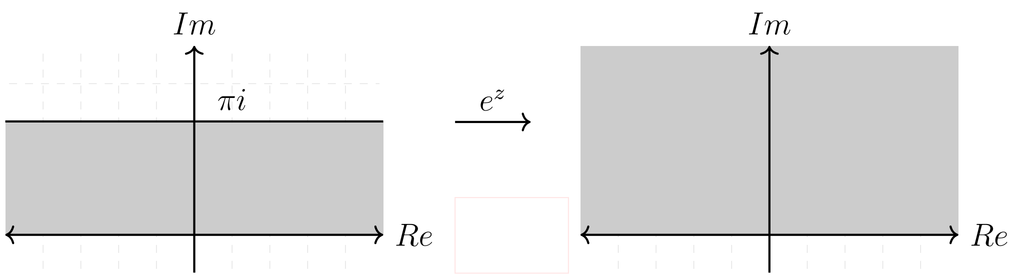
And

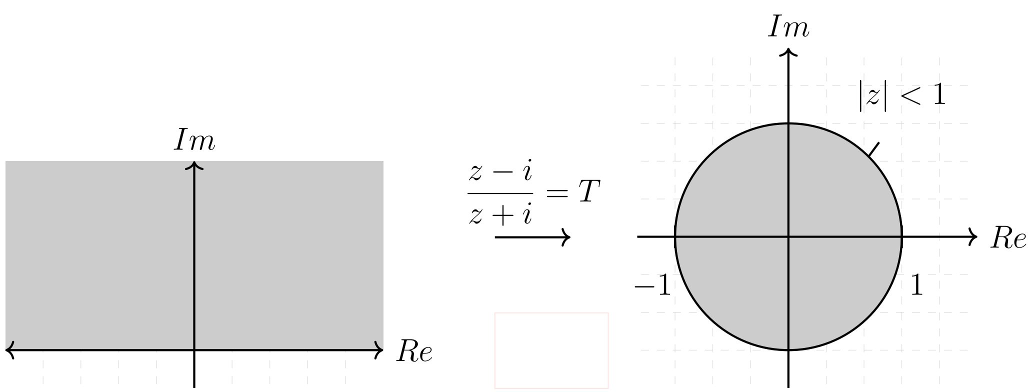
$$T(z) = \frac{z - w_3}{z - w_4} : \frac{w_2 - w_3}{w_2 - w_4}$$

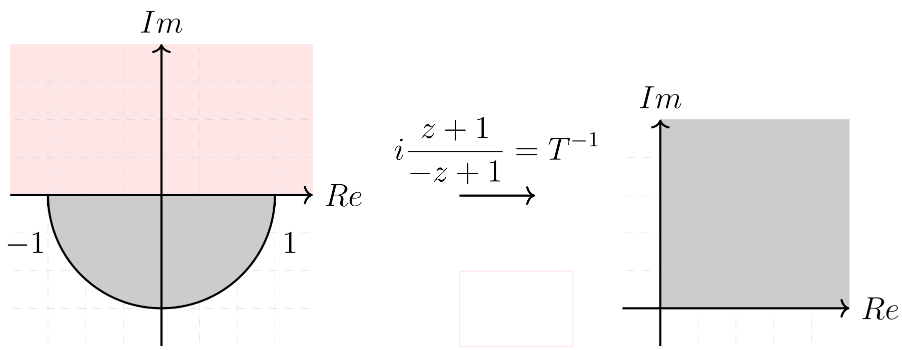
is the unique transformation sending $w_2 \mapsto 1$, $w_3 \mapsto 0$, $w_4 \mapsto \infty$.

Formula 3. *Conformal Maps*









- z^2 doubles angles,
- $\sqrt{z}$ halves angles,
- iz rotates $90°$ counterclockwise