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Complex Analysis Exam Spring 2018

Problem 1. Show that there is no holormophic function f in $\mathbb{D} = \{|z| < 1\}$ so that $|f(z_n)| \to \infty$ whenever $|z_n| \to 1$ ($z_n \in \mathbb{D}$).

Solution. Let f be holomorphic in $\mathbb{D}$. Assume $|f(z_n)| \to \infty$ whenever $|z_n| \to 1$. Since zeros are isolated, there at most countably many inside $\mathbb{D}$.

However, if there are infinitely many zeros, then there cannot exist an accumulation point inside $\mathbb{D}$, otherwise $f \equiv 0$ by the identity theorem. But then there would exist a limit point inside $\overline{\mathbb{D}}$ and since the limit point is not in $\mathbb{D}$, its on the boundary. Thus, the sequence of zeros of f $\{a_n\}$, satisfies that $|a_n| \to 1$ and $|f(a_n)| \to 0$, contradicting the hypothesis.

Therefore, f can have only finitely many zeros inside $\mathbb{D}$.

Let $f(z) = (z - a_1) \cdots (z - a_k)g(z)$ for $g(z)$ analytic in $\mathbb{D}$ and never zero. Then

$$\frac{f(z)}{(z - a_1) \cdots (z - a_k)} = g(z)$$

which is holomorphic and nonzero. Thus, $\frac{1}{g}$ is holomorphic and nonzero.

Since $a_i \in \mathbb{D}$, for any sequence $\{z_n\}$ with $|z_n| \to 1$, for all $\varepsilon > 0$, there exists N so $|z_n - a_i| \geq \varepsilon$ for all $n \geq N$. Namely, since f explodes near the boundary of $\mathbb{D}$, so too must g.

Therefore, $\frac{1}{|g(z_n)|} \to 0$ whenever $|z_n| \to 1$ so by the maximum modulus principle, implies $\frac{1}{g} \equiv 0$. Clearly a contradiction.

Thus, g cannot exist and so neither can f. ✌

Problem 2. Assume that f is analytic in the unit disk $\mathbb{D} = \{z : |z| < 1\}$. Prove that f is odd if and only if all the terms in the Taylor series for f at $z_0 = 0$ have only odd powers.

Solution.

$\boxed{\Longrightarrow}$ Assume f is odd. Then $f(-z) = -f(z)$ for all $z \in \mathbb{D}$. Since f is analytic, it has a taylor series at 0. Thus,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad z \in \mathbb{D}.$$

Now, for $r > 0$ small, we get that

$$\begin{aligned}
a_{2n} &= \frac{f^{2n}(0)}{(2n)!} \\
&= \frac{1}{2\pi i}\int_{|z|=r} \frac{f(z)}{z^{2n+1}}dz \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{f(re^{i\theta})}{r^{2n}e^{2ni\theta}}d\theta \qquad z = re^{i\theta} \\
&= \frac{1}{2\pi}\left[\int_0^{\pi} \frac{f(re^{i\theta})}{r^{2n}e^{2ni\theta}} + \int_{\pi}^{2\pi} \frac{f(re^{i\theta})}{r^{2n}e^{2ni\theta})}\right] \\
&= \frac{1}{2\pi}\left[\int_0^{\pi} \frac{f(re^{i\theta})}{r^{2n}e^{2ni\theta}} + \int_0^{\pi} \frac{f(re^{i(\theta-\pi)})}{r^{2n}e^{2ni(\theta-\pi)}}\right] \\
&= \frac{1}{2\pi}\left[\int_0^{\pi} \frac{f(re^{i\theta})}{r^{2n}e^{2ni\theta}} + \int_0^{\pi} \frac{f(-re^{i\theta})}{r^{2n}e^{2ni\theta}}\right] \\
&= \frac{1}{2\pi}\left[\int_0^{\pi} \frac{f(re^{i\theta})}{r^{2n}e^{2ni\theta}} + \int_0^{\pi} \frac{-f(re^{i\theta})}{r^{2n}e^{2ni\theta}}\right] \\
&= 0
\end{aligned}$$

for all n since f is odd.

Thus,

$$f(z) = \sum_{n=0}^{\infty} a_{2n+1} z^{2n+1}.$$

$\boxed{\Longleftarrow}$ This clear. Since f is analytic, it has a taylor series at 0. Thus,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_{2n+1} z^{2n+1}$$

since all the terms in the taylor series have only odd powers.

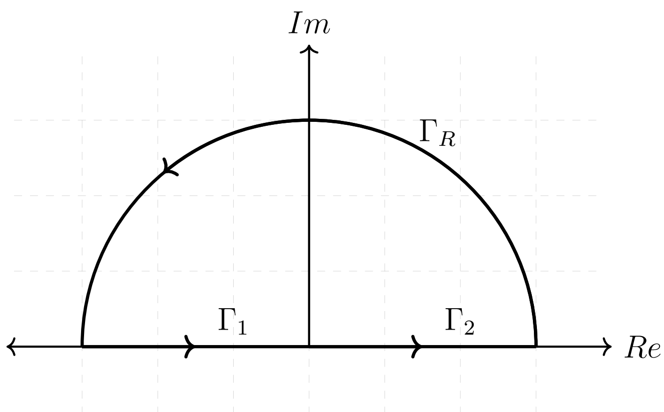
Thus,

$$f(-z) = \sum_{n=0}^{\infty} a_{2n+1}(-z)^{2n+1} = \sum_{n=0}^{\infty}(-1)^{2n+1}a_{2n+1}z^{2n+1} = -\sum_{n=0}^{\infty} a_{2n+1}z^{2n+1} = -f(z).$$

So f is odd. ✌

Problem 3. Evaluate
$$\int_{-\infty}^{\infty} \frac{\cos(2x)}{x^2+1}dx$$

Solution. We examine $\int \frac{e^{i2z}}{z^2+1}dz$ and use "Ol' Faithful" with the origin. Namely, the contour around the upper half plane.



Let

$$I_1 = \int_{\Gamma_1} \frac{e^{2iz}}{z^2+1}dz$$
$$I_2 = \int_{\Gamma_2} \frac{e^{2iz}}{z^2+1}dz$$
$$I_R = \int_{\Gamma_R} \frac{e^{2iz}}{z^2+1}dz$$

Now, we note that $\cos(-2x) = \cos(2x)$ since cos is an even real function and so

$$\int_{-\infty}^{\infty} \frac{\cos(2x)}{x^2+1}dx = 2\int_{0}^{\infty} \frac{\cos(2x)}{x^2+1}dx$$

Then, we note that

$$\begin{aligned}
I_1 + I_2 &= \int_{\Gamma_1} \frac{e^{2iz}}{z^2+1}dz + \int_{\Gamma_2} \frac{e^{2iz}}{z^2+1}dz \\
&= \int_{-R}^{0} \frac{e^{2ix}}{x^2+1}dx + \int_{0}^{R} \frac{e^{2ix}}{x^2+1}dx \\
&= \int_{R}^{0} -\frac{e^{-2ix}}{x^2+1}dx + \int_{0}^{R} \frac{e^{2ix}}{x^2+1}dx \\
&= \int_{0}^{R} \frac{e^{-2ix}}{x^2+1}dx + \int_{0}^{R} \frac{e^{2ix}}{x^2+1}dx \\
&= \int_{0}^{R} \frac{e^{2ix}+e^{-2ix}}{x^2+1}dx \\
&= \int_{0}^{R} \frac{2\cos(2x)}{x^2+1}dx \\
&= 2\int_{0}^{R} \frac{\cos(2x)}{x^2+1}dx
\end{aligned}$$

Finally,

$$\begin{aligned}
|I_R| &= \left|\int_{\Gamma_R} \frac{e^{2iz}}{z^2+1}dz\right| \\
&\leq \int_{\Gamma_R} \frac{|e^{2iz}|}{|z^2+1|}d|z| \\
&\leq \int_{\Gamma_R} \frac{e^{-2\Im\mathrm{m}(z)}}{|z|^2-1}d|z| \\
&\leq \int_{0}^{\pi} \frac{Re^{-2R\sin\theta}}{R^2-1}d\theta \qquad z = Re^{i\theta} \\
&= \frac{R}{R^2-1}\int_{0}^{\pi} e^{-2R\sin\theta}d\theta \\
&\leq \frac{R}{R^2-1}\int_{0}^{\pi} d\theta \qquad \sin\theta \geq 0 \text{ on upper half plane} \\
&= \frac{\pi R}{R^2-1} \to 0 \qquad R \to \infty
\end{aligned}$$

Therefore, by the residue theorem,

$$\begin{aligned}
2\pi i \mathrm{Res}_{z=i}\frac{e^{2iz}}{z^2+1} &= 2\pi i \frac{e^{-2}}{i+i} \\
&= \frac{\pi}{e^2} \\
&= \lim_{R\to 0}(I_1 + I_2 + I_R) \\
&= 2\int_{0}^{\infty} \frac{\cos(2x)}{x^2+1}dx \\
\int_{-\infty}^{\infty} \frac{\cos(2x)}{x^2+1}dx &= \frac{\pi}{e^2}
\end{aligned}$$

Problem 4. Let $\mathbb{D} = \{|z| < 1\}$ be the open unit disk and $\overline{\mathbb{D}}$ its closure. Let $f : \mathbb{D} \to \mathbb{C}$ be analytic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. Assume that f only takes real values on $\partial\overline{\mathbb{D}} = \{|z| = 1\}$. Prove that f is constant.

Solution. Now, since f is analytic $\mathfrak{Im}(f(z))$ is a harmonic function in the disk which is continuous ont he boundary of the unit disk. Thus, by the maximimum modulus principle, $|\mathfrak{Im}(f(z))|$ has to reach a maximum on $\partial\overline{\mathbb{D}}$.

However,

$$\begin{aligned}
|\mathfrak{Im}(f(z))| &\leq \sup_{z\in\partial\overline{\mathbb{D}}} |\mathfrak{Im}(f(z))| \\
&= \sup_{z\in\partial\overline{\mathbb{D}}} |\frac{1}{2}(f(z) - \overline{f(z)})| \\
&= \sup_{z\in\partial\overline{\mathbb{D}}} |\frac{1}{2}(f(z) - f(z))| \qquad f(z) \in \mathbb{R} \text{ whenever } z \in \partial\overline{\mathbb{D}} \\
&= 0
\end{aligned}$$

Thus, f is a real function.

However, Cauchy-Riemann then implies that $f = u + iv = u$ must be a constant since u must be a constant because $u_x = v_y = 0$ and $u_y = -v_x = 0$.

Thus, f is a real constant. ✌