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Complex Analysis Exam Fall 2018

> **Problem 1.** Let $a > 0$. Compute
> $$\int_0^\pi \frac{d\theta}{a^2 + \sin^2\theta}$$

Solution. Since $\sin^2\theta = \sin^2(\theta + \pi)$, we have that

$$\int_0^{2\pi} \frac{d\theta}{a^2 + \sin^2\theta} = 2\int_0^\pi \frac{d\theta}{a^2 + \sin^2\theta}$$

and so

$$\begin{aligned}
\int_0^\pi \frac{d\theta}{a^2 + \sin^2\theta} &= \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{a^2 + \sin^2\theta} \\
&= \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{a^2 + \frac{1-\cos\theta}{2}} \\
&= \frac{1}{2}\int_0^{2\pi} \frac{2d\theta}{2a^2 + 1 - \frac{e^{i\theta}+e^{-i\theta}}{2}} \\
&= \frac{1}{2}\int_0^{2\pi} \frac{4d\theta}{4a^2 + 2 - e^{i\theta} - e^{-i\theta}} \\
&= \frac{1}{2}\int_0^{2\pi} \frac{-4e^{i\theta}d\theta}{e^{2i\theta} - (4a^2+2)e^{i\theta} + 1} \\
&= \frac{1}{2}\int_{\{|z|=1\}} \frac{4idz}{z^2 - (4a^2+2)z + 1} \qquad z = e^{i\theta} \\
&= \frac{4i}{2}\int_{\{|z|=1\}} \frac{dz}{(z-\alpha)(z-\beta)} && (1) \\
&= 2i\left(2\pi i \text{Res}_{z=\beta}\frac{1}{(z-\alpha)(z-\beta)}\right) && (2) \\
&= -4\pi\frac{1}{\beta - \alpha} \\
&= \frac{-4\pi}{-4a\sqrt{a^2+1}} \\
&= \frac{\pi}{a\sqrt{a^2+1}}
\end{aligned}$$

with (1) since

$$\begin{aligned}
x^2 - (4a^2+2)x + 1 = 0 \implies x &= \frac{4a^2+2 \pm \sqrt{(4a^2+2)^2-4}}{2} \\
&= \frac{4a^2+2 \pm \sqrt{16a^4+16a^2+4-4}}{2} \\
&= \frac{4a^2+2 \pm \sqrt{16a^4+16a^2}}{2} \\
&= \frac{4a^2+2 \pm 4a\sqrt{a^2+1}}{2} \\
&= 2a^2+1 \pm 2a\sqrt{a^2+1} \\
\alpha &= 2a^2+1+2a\sqrt{a^2+1} \\
\beta &= 2a^2+1-2a\sqrt{a^2+1}
\end{aligned}$$

and (2) because $\alpha > 1$ for all $a > 0$. ✌

Problem 2. Find the number of solutions of the equation $z - 2 - e^{-z} = 0$ in $H = \{z \in \mathbb{C} : \mathcal{Re}(z) > 0\}$.

Solution. Let $f(z) = z - 2 - e^{-z}$. Then

$$f(iy) = iy - 2 - e^{-iy} = iy - 2 - \cos(y) - i\sin(y) = -2 - \cos(y) + i(y - \sin(y))$$

and so $\mathcal{Re}(f(iy)) \leq -1 < 0$ for all $y \in \mathbb{R}$. Thus, f sends the imaginary axis to the left-half plane, away from the origin.

Now, if $z - 2 - e^{-z} = 0$ then

$$|e^{-z}| = e^{\mathcal{Re}(-z)} = |z - 2| \geq |z| - 2 > 1 \qquad \text{for all } |z| > 3$$

and so for if z is a root of f and $|z| > 3$, then $\mathcal{Re}(-z) = -\mathcal{Re}(z) > 0$ and so $\mathcal{Re}(z) < 0$ and z lies in the left-half plane. Namely, the total change in argument is at most π.

Now, let $z = Re^{i\theta}$ for $\theta \in (-\pi/2, \pi/2)$.

Then
$$\frac{1}{R} f(Re^{i\theta}) = e^{i\theta} - \frac{2}{R} - \frac{e^{-Re^{i\theta}}}{R} \to e^{i\theta} \qquad R \to \infty$$
and so by the argument principle, f has a total change in argument of π.

Namely, f can have at most one zero in the right half plane.

Since $f(0) < 0$ and $f(10) = 8 - \frac{1}{e^{10}} > 7 > 0$ by the intermediate value theorem, f has a zero on the positive real axis.

Namely, f has exactly one zero in the right-half plane.

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Problem 3. Let $\Omega \neq \mathbb{C}$ be simply connected and let for any $c \in \Omega$, the mapping $\varphi_c : \Omega \to \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be conformal so that $\varphi_c(c) = 0$. Let $g_c(z) = \log|\varphi_c(z)|$, $z \in \Omega \backslash \{c\}$.

Show that $g_a(b) = g_b(a)$ for any distinct $a, b \in \Omega$.

Solution. Let $a, b \in \Omega$ be distinct and let

$$\begin{aligned} T : \mathbb{D} &\to \mathbb{D} \\ z &\mapsto \frac{z - \varphi_a(b)}{1 - \overline{\varphi_a(b)} z} \end{aligned}$$

Then T is a conformal map and isomorphism of the unit disk.

Note that φ_c being conformal for all c, implies that φ_c is locally invertible by the inverse function theorem.

Now, we examine $f(z) = (T \circ \varphi_a \circ \varphi_b^{-1})(z)$.

Then $f : \mathbb{D} \to \mathbb{D}$ with

$$f(0) = T(\varphi_a(\varphi_b^{-1}(0))) = T(\varphi_a(b)) = 0.$$

Therefore, by Schwarz' Lemma,

$$|f(z)| \leq |z| \qquad z \in \mathbb{D}$$

and so namely,

$$\begin{aligned} |f(\varphi_b(a))| &= |T(\varphi_a(\varphi_b^{-1}(\varphi_b(a))))| \\ &= |T(\varphi_a(a))| \\ &= |T(0)| \\ &= |\varphi_a(b)| \\ &\leq |\varphi_b(a)| \end{aligned}$$

Similarly, we can show using a different isomorphism of the disk that

$$|\varphi_b(a)| \leq |\varphi_a(b)|$$

and so

$$|\varphi_a(b)| = |\varphi_b(a)| \implies g_a(b) = g_b(a).$$

✌

Problem 4. Let $a \in \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and

$$f_a(z) = \frac{a - z}{1 - \overline{a}z}, \qquad z \in \overline{\mathbb{D}}.$$

Show that f_a is a holomorphic bijective mapping of $\mathbb{D}$ onto $\mathbb{D}$ which is its own inverse.

Solution. f is a Mobius Transform, and so namely it is bijective with inverse

$$f_a^{-1}(z) = \frac{z - a}{\overline{a}z - 1} = \frac{a - z}{1 - \overline{a}z} = f_a(z).$$

Note that being able to write the inverse directly implies that f is bijective.

Since $|a| < 1$, $\left|\frac{1}{\overline{a}}\right| = \frac{1}{|a|} > 1$ and so f_a does not have any poles inside the unit disk. Namely, f_a is holomorphic, bijective, and analytic in the unit disk. ✌