Kayla Orlinsky

Complex Analysis Exam Spring 2017

Problem 1. Let γ be a circle with radius 1 and center 0 with positive direction of integration. Compute

$$\int_\gamma e^{1/z} dz$$

Solution.

$$\begin{aligned}
\int_{|z|=1} e^{1/z} dz &= \int_{\{} |z| = 1\} \sum_{n=0}^{\infty} \frac{1}{z^n n!} dz \\
&= \sum_{n=0}^{\infty} \int_{|z|=1} \frac{1}{z^n n!} dz && \text{since both the integral and sums exist} \\
&= \sum_{n=1}^{\infty} \int_{|z|=1} \frac{1}{z^n n!} dz && \text{first integral is of an analytic function so it dies} \\
&= \sum_{n=1}^{\infty} \left(\frac{2\pi i f^{(n-1)}(0)}{(n!)^2} \right) && f(z) = 1 \\
&= 2\pi i
\end{aligned}$$

✌

Problem 2. Assume that f is a holomorphic function on the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ satisfying

$$f(z)^3 = \overline{f(z)} \qquad z \in D.$$

Prove that f is constant.

Solution.

$$f^4(z) = \overline{f(z)}f(z) = |f(z)|^2 \implies f^2(z) = |f(z)| \in \mathbb{R} \qquad z \in D.$$

Thus, for each $z \in D$, $f(z)$ must be either purely real or purely imaginary. Namely,

$$f(D) \subset \{x : x \in \mathbb{R}\} \bigcup \{iy : y \in \mathbb{R}\}.$$

However, by the open mapping theorem, if f is nonconstant, then $f(D)$ must be an open subset of the plane, since f is holomorphic. Since subsets of the real and imaginary lines are not open in $\mathbb{C}$, f must be constant.

Alternatively,

$$|f(z)|^3 = |\overline{f(z)}| = |f(z)| \implies |f(z)|^3 = |f(z)| \implies |f(z)| = 0 \text{ or } 1 \forall z \in \mathbb{D}.$$

Thus, f is either identically 0 (constant) or f sends the open disk to the unit circle which contradicts the open mapping theorem.

✌

Problem 3. Let $f(z) = \sum_{n=1}^{\infty} z^{n!}$.

(a) Show that $f(z)$ is holomorphic in the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$.

(b) Show that f does not have any holomorphic extension, that is, there exists no g holomorphic on some open set $U \supset D$ such that $U \neq D$ and $f = g|_D$. (Hint: Consider $e^{i\theta}$ where θ is rational.)

Solution. This question is nearly identical to **Spring 2015: Problem 2**. Thus, the proof here is the same as the one given there.

(a) f is holormophic if and only if the sum converges uniformly. For $|z| < r < 1$, we have that $|z|^{n!} \leq r^{n!} \leq r^n$ since $r < 1$, and so f converges uniformly by Weierstrass M-test on $\{|z| < r\}$. Since this holds for all $r < 1$, we have that f converges uniformly and is therefore analytic on D.

(b) Let $D \subsetneq U$ open. Since $U \neq D$ and U is open, U must contain some z_0 with $|z_0| = 1$.

Again, since U is open, any neighborhood of z_0 must contain some $z = e^{i\pi/m}$ for $m \in \mathbb{Z}$ since $|z| = 1$.

However, then

$$f(z) = \sum_{n=0}^{m-1} e^{in!\pi/m} + \sum_{n=m}^{\infty} e^{i\pi n!/m} = \sum_{n=0}^{m-1} e^{in!\pi/m} + \sum_{n=m}^{\infty} 1 = \infty$$

since $n!/m \in \mathbb{Z}$ for $n \geq m$.

Now, f has a non-removable singularity in every neighborhood of z_0.

Namely, f cannot be extended to a neighborhood to z_0 since it will not converge in any punctured neighborhood of z_0.

✌

Problem 4. Let f be an entire function such that

$$f(z+m+ni) = f(z), \qquad z \in \mathbb{C} \qquad m, n \in \mathbb{Z}.$$

Prove that f is a constant function.

Solution. WLOG, let $f(0) = 0$, else we can let $g(z) = f(z) - f(0)$.

Let $z \in \mathbb{C}$. Then let m be the integer component of $\mathfrak{Re}(z)$ and n be the integer part of $\mathfrak{Im}(z)$. Then for some $\alpha, \beta \in (0,1)$, we can write

$$z = (m+\alpha) + (n+\beta)i \implies f(z) = f((\alpha+\beta i) + (m+ni)) = f(\alpha+\beta i).$$

Namely, f is uniquely determined by the values in takes in $\mathbb{D} = \{|z| < 1\}$. Since f is entire, by the maximum modulus principle, f must attain a maximum on $\partial\overline{\mathbb{D}} = \{|z| = 1\}$.

Now, because f is entire, it has no poles, so

$$M = \sup_{z \in \partial\overline{\mathbb{D}}} |f(z)| < \infty$$

and

$$|f(z)| \leq M \qquad \text{for all } z \in \mathbb{C}$$

so by Louiville's f is entire and bounded so it is constant. ✌