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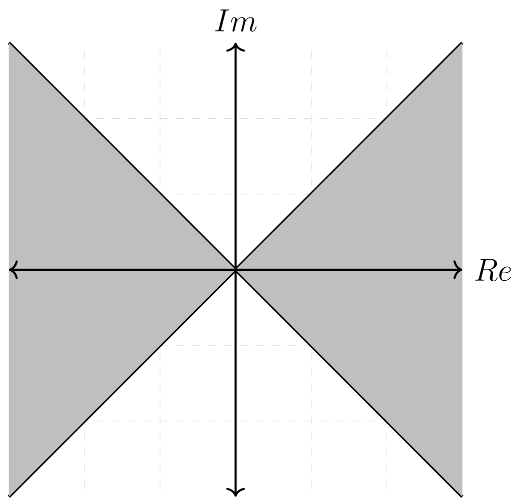
Complex Analysis Exam Fall 2017

> **Problem 1.** Let $f(z) = u(z)+iv(z)$ be an entire function and assume that $|u(z)| \geq |v(z)|$ for all $z \in \mathbb{C}$. Show that f is a constant.

Solution. Clearly $f(\mathbb{C}) \subset \{x + iy : |y| \leq |x|\}$. This set can be broken down into the four quadrants to get



Now, since $\mathbb{C}$ is open, by the open mapping theorem, $f(\mathbb{C})$ must be open. Therefore, $f(\mathbb{C})$ cannot contain 0.

Furthermore, $f(\mathbb{C})$ must be connected. Thus, $f(\mathbb{C})$ is either entirely in the left-half plane or entirely in the right-half plane.

WLOG, assume $\mathcal{R}\text{e}(f(z)) > 0$ for all z. Namely, that f is contained in the right half plane.

Then, let $g(z) = iz$ and $T(z) = \frac{z-i}{z+i}$.

Then $T \circ g \circ f$ is an entire map from $\mathbb{C}$ to the unit disk $\mathbb{D}$. Therefore, by Louiville's Theorem, $T \circ g \circ f$ is constant.

However, this forces f to be constant since g and T certainly are not constant. ✌

Problem 2. Let $\alpha \in (0,1)$ and $n \in \mathbb{N}$. Prove that the equation $e^z(z-1)^n = \alpha$ has exactly n *simple* roots in the right half plane $\{z : \Re\text{e}(z) > 0\}$.

Solution. This problem is nearly identical to **Spring 2016: Problem 2**, with the only difference being that the root is now a real number.

First, we note that if $e^z(z-1)^n = \alpha$ and z is in the right half plane, then $\Re\text{e}(z) \geq 0$ so $|e^z| \geq 1$. Namely,

$$\begin{aligned} |z-1|^n &\leq |e^z||z-1|^n \\ &= |\alpha| \\ &< 1 \end{aligned}$$

Thus, $|z-1| < 1$ and so the roots that lie in the right half plane are all contained in the circle $\{|z-1| < 1\}$.

Now, let $f(z) = e^z(z-1)^n - \alpha$ and $g(z) = e^z(z-1)^n$.

Then, in the circle $\{|z-1| \leq 1\}$, g has exactly n roots (since multiplicities are counted).

Furthermore, on $\{|z-1| = 1\}$, we have that

$$|f(z) - g(z)| = |\alpha| < 1 = |z-1| \leq |g(z)|$$

and so f and g have the same number of roots inside the circle $\{|z-1| < 1\}$ which is n.

Now, if f has any non-simple roots (any repeated roots), then f and f' would have a root in common.

However,

$$f'(z) = e^z(z-1)^n + ne^z(z-1)^{n-1} = e^z(z-1)^{n-1}[z-1+n]$$

and so if f and f' are simultaneously zero, then

$$\begin{aligned} f(z) &= 0 \\ e^z(z-1)^n &= \alpha \\ e^z(z-1)^{n-1} &= \frac{\alpha}{z-1} \\ f'(z) &= 0 \\ e^z(z-1)^{n-1}[z-1+n] &= 0 \\ \frac{\alpha}{z-1}[z-1+n] &= 0 \end{aligned}$$

since $z = 1$ is not a root of f since $\alpha \neq 0$ and $z - 1 = -n$ is not in $\{|z-1| < 1\}$ for any n, f' and f share no roots in $\{|z-1| < 1\}$. And since we have already shown that all of the roots of f in the right half plane lie in this circle, all n roots of f in the right-half plane have multiplicity 1 and so are simple. ✌

Problem 3. Evaluate the integral

$$\int_0^{2\pi} \frac{dt}{\cos t - 2}.$$

Solution. By the Residue Theorem,

$$\begin{aligned}
\int_0^{2\pi} \frac{dt}{\cos t - 2} &= \int_0^{2\pi} \frac{dt}{\frac{e^{it}+e^{-it}}{2} - 2} \\
&= \int_0^{2\pi} \frac{2e^{it}dt}{e^{2it} - 4e^{it} + 1} \\
&= \int_{|z|=1} \frac{-2idz}{z^2 - 4z + 1} \qquad z = e^{it} \\
&= \int_{|z|=1} \frac{-2idz}{(z-(2+\sqrt{3}))(z-(2-\sqrt{3}))} \qquad (1) \\
&= 2\pi i \mathrm{Res}_{z=2-\sqrt{3}} \frac{-2i}{(z-(2+\sqrt{3}))(z-(2-\sqrt{3}))} \\
&= 4\pi \frac{1}{2-\sqrt{3}-(2+\sqrt{3})} \\
&= 4\pi \frac{1}{-2\sqrt{3}} \\
&= \frac{-2\pi}{\sqrt{3}}
\end{aligned}$$

With (1) since

$$\begin{aligned}
z^2 - 4z + 1 &= 0 \\
z &= \frac{4 \pm \sqrt{16-4}}{2} \\
&= \frac{4 \pm 2\sqrt{3}}{2} \\
&= 2 \pm \sqrt{3}
\end{aligned}$$

and (2) since $|2+\sqrt{3}| > 1$ and $|2-\sqrt{3}| < 1$. ✌

Problem 4. Write an entire function which has the simple zeros $1, 4, 9, 16, 25, ...$ and has no other zeros.

Solution. The zeros are $1^2, 2^2, 3^2, 4^2, 5^2, ...$ Namely, if such a function exists then it is of the form

$$f(z) = \prod_{n=1}^{\infty} \frac{(z - n^2)}{n^2} = \prod_{n=1}^{\infty} \left(\frac{z}{n^2} - 1 \right) = - \prod_{n=1}^{\infty} \left(1 - \frac{z}{n^2} \right)$$

If we can show that the product converges uniformly, then f will be analytic everywhere.

Since $\prod(1 - a_n)$ converges absolutely and uniformly if and only if $\sum a_n$ converges absolutely and uniformly.

However, clearly,

$$\sum_{n=1}^{\infty} \frac{|z|}{n^2} < \infty$$

for all z, and since for $\{|z| < M\}$ we get that $\frac{|z|}{n^2} < \frac{M}{n^2}$ and so the sum converges uniformly.

Since this holds for all M, the sum converges uniforly everywhere and so f defines an analytic function whose only zeros are $\{n^2\}_1^\infty$ as desired. ✌