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Complex Analysis Exam Spring 2016

Problem 1. Let $a \in \mathbb{C}$ be such that $0 < |a| < 1$, and set

$$f(z) = \frac{1 - z^2}{z^2 - (a + \frac{1}{a})z + 1}.$$

Find the Laurent expansion of f in a neighborhood of the unit circle $|z| = 1$.

Solution. We interpret "A neighborhood of the unit circle" to actually mean for some $B_r(0) \subset \mathbb{D}$.

First,

$$\begin{aligned} z^2 - (a + \frac{1}{a})z + 1 = 0 \\ \implies z &= \frac{a + \frac{1}{a} \pm \sqrt{(a + \frac{1}{a})^2 - 4}}{2} \\ &= \frac{a + \frac{1}{a} \pm \sqrt{a^2 + 2 + \frac{1}{a^2} - 4}}{2} \\ &= \frac{a + \frac{1}{a} \pm \sqrt{a^2 - 2 + \frac{1}{a^2}}}{2} \\ &= \frac{a + \frac{1}{a} \pm \sqrt{(a - \frac{1}{a})^2}}{2} \\ &= \frac{a + \frac{1}{a} \pm (a - \frac{1}{a})}{2} \\ &= a, \frac{1}{a} \end{aligned}$$

Thus,

$$f(z) = \frac{(1 - z^2)}{(z - a)(z - \frac{1}{a})}.$$

Now,

$$f(z) = (1-z^2)\left[\frac{\frac{a}{a^2-1}}{z-a} - \frac{\frac{a}{a^2-1}}{z-\frac{1}{a}}\right]$$
$$= (1-z^2)\left[\frac{\frac{a}{a^2-1}}{-a(1-\frac{z}{a})} - \frac{\frac{a}{a^2-1}}{\frac{1}{a}(1-az)}\right]$$
$$= (1-z^2)\left[\frac{\frac{a^2}{a^2-1}}{1-az} - \frac{\frac{1}{a^2-1}}{1-\frac{z}{a}}\right]$$

Now, for $|z| < |a|$, we have that $|az| < |a|^2 < 1$ and $\frac{|z|}{|a|} < 1$.

Thus, on $\{|z| < |a|\}$,

$$\frac{\frac{a^2}{a^2-1}}{1-az} = \frac{a^2}{a^2-1}\sum_{n=0}^{\infty}(az)^n$$

and

$$\frac{\frac{1}{a^2-1}}{1-\frac{z}{a}} = \frac{1}{a^2-1}\sum_{n=0}^{\infty}\left(\frac{z}{a}\right)^n$$

so

$$f(z) = (1-z^2)\sum_{n=0}^{\infty}\left[\frac{a^2}{a^2-1}a^n - \frac{a^{-n}}{a^2-1}\right]z^n$$
$$= (1-z^2)\sum_{n=0}^{\infty}\left[\frac{a^{n+2}-a^{-n}}{a^2-1}\right]z^n$$
$$= (1-z^2)\sum_{n=0}^{\infty}\alpha_n z^n \qquad \alpha_n = \frac{a^{n+2}-a^{-n}}{a^2-1}$$
$$= \sum_{n=0}^{\infty}\alpha_n z^n - \sum_{n=0}^{\infty}\alpha_n z^{n+2}$$
$$= \sum_{n=0}^{\infty}\alpha_n z^n - \sum_{k=-2}^{\infty}\alpha_{k+2} z^k$$
$$= \sum_{n=0}^{\infty}\alpha_n z^n - \alpha_0 z^{-2} - \alpha_1 z^{-1} - \sum_{k=0}^{\infty}\alpha_{k+2} z^k$$
$$= -\alpha_0 z^{-2} - \alpha_1 z^{-1} + \sum_{n=0}^{\infty}\alpha_n z^n - \sum_{n=0}^{\infty}\alpha_{n+2} z^n$$
$$= -\alpha_0 z^{-2} - \alpha_1 z^{-1} + \sum_{n=0}^{\infty}(\alpha_n - \alpha_{n+2}) z^n$$
$$= -z^{-2} - \frac{a^2+1}{a}z^{-1} + \sum_{n=0}^{\infty}(\alpha_n - \alpha_{n+2}) z^n$$

Alternatively, note that we could have interpreted "A neighborhood of the unit circle" to actually mean that we must find the Laurent expansion in some annulus $\{s < |z| < t\}$ with $s < 1 < t$ so the unit circle is contained in the annulus.

In this case, we would use that for $|a| < |z| < \frac{1}{|a|}$, so $\frac{|a|}{|z|} < 1$ and $|az| < 1$.

Then we would simply rewrite $f(z)$ to be

$$\begin{aligned} f(z) &= (1-z^2)\left[\frac{\frac{a}{a^2-1}}{z-a} - \frac{\frac{a}{a^2-1}}{z-\frac{1}{a}}\right] \\ &= (1-z^2)\left[\frac{\frac{a}{a^2-1}}{z(1-\frac{a}{z})} - \frac{\frac{a}{a^2-1}}{\frac{1}{a}(1-az)}\right] \\ &= (1-z^2)\left[\frac{\frac{1}{z}\frac{a}{a^2-1}}{1-\frac{a}{z}} - \frac{\frac{a^2}{a^2-1}}{1-az}\right] \end{aligned}$$

and so on $\{|a| < |z| < \frac{1}{|a|}\}$, which is an annulus containing the unit circle since $|a| < 1 < \frac{1}{|a|}$,

$$\frac{\frac{1}{z}\frac{a}{a^2-1}}{1-\frac{a}{z}} = \frac{a}{a^2-1}\sum_{n=0}^{\infty}\frac{a^n}{z^{n+1}}$$

and

$$\frac{\frac{a^2}{a^2-1}}{1-az} = \frac{a^2}{a^2-1}\sum_{n=0}^{\infty}(az)^n$$

$$\begin{aligned} f(z) &= (1-z^2)\left[\frac{a}{a^2-1}\sum_{n=0}^{\infty}\frac{a^n}{z^{n+1}} - \frac{a^2}{a^2-1}\sum_{n=0}^{\infty}(az)^n\right] \\ &= \frac{a}{a^2-1}\left[\sum_{n=0}^{\infty}\frac{a^n}{z^{n+1}} - \sum_{n=0}^{\infty}\frac{a^n}{z^{n-1}}\right] - \frac{a^2}{a^2-1}\left[\sum_{n=0}^{\infty}a^n z^n - \sum_{n=0}^{\infty}a^n z^{n+2}\right] \\ &= \frac{a}{a^2-1}\left[\sum_{n=0}^{\infty}\frac{a^n}{z^{n+1}} - z - a - \sum_{n=2}^{\infty}\frac{a^n}{z^{n-1}}\right] - \frac{a^2}{a^2-1}\left[1 + az + \sum_{n=2}^{\infty}a^n z^n - \sum_{n=0}^{\infty}a^n z^{n+2}\right] \\ &= \frac{a}{a^2-1}\left[\sum_{n=0}^{\infty}\frac{a^n}{z^{n+1}} - z - a - \sum_{k=0}^{\infty}\frac{a^{k+2}}{z^{k+1}}\right] - \frac{a^2}{a^2-1}\left[1 + az + \sum_{k=0}^{\infty}a^{k+2}z^{k+2} - \sum_{n=0}^{\infty}a^n z^{n+2}\right] \\ &= \frac{a}{a^2-1}\left[-z - a + \sum_{n=0}^{\infty}\frac{a^n - a^{n+2}}{z^{n+1}}\right] - \frac{a^2}{a^2-1}\left[1 + az + \sum_{n=0}^{\infty}(a^{n+2}-a^n)z^{n+2}\right] \\ &= -\frac{a}{a^2-1}z - \frac{a^2}{a^2-1} + \frac{a}{a^2-1}\sum_{n=0}^{\infty}\frac{a^n - a^{n+2}}{z^{n+1}} - \frac{a^2}{a^2-1} - \frac{a^3}{a^2-1}z - \frac{a^2}{a^2-1}\sum_{n=0}^{\infty}(a^{n+2}-a^n)z^{n+2} \\ &= -\frac{2a^2}{a^2-1} - \frac{a+a^3}{a^2-1}z + \frac{a}{a^2-1}\sum_{n=0}^{\infty}\frac{a^n - a^{n+2}}{z^{n+1}} - \frac{a^2}{a^2-1}\sum_{n=0}^{\infty}(a^{n+2}-a^n)z^{n+2} \end{aligned}$$

✌

Problem 2. Let $a \in \mathbb{C}$ be such that $0 < |a| < 1$ and let $n \in \mathbb{N}$. Show that $e^z(z-1)^n = a$ has exactly n simple roots in the half-plane $\{z \in \mathbb{C} : \mathcal{Re}(z) > 0\}$.

Solution. First, note that if z is in the right-half plane and $e^z(z-1)^n = a$ then

$$\begin{aligned} |z-1|^n &< |e^z||z-1|^n \qquad |e^z| = e^{\mathcal{Re}(z)} > e^0 = 1 \\ &= |a| < 1 \\ \implies |z-1| &< 1 \end{aligned}$$

so if $f(z) = e^z(z-1)^n - a$ has a zero in the right half plane, it is in the disk $\{|z-1| < 1\}$.

Now, let $g(z) = e^z(z-1)^n$. Then g has n zeros in the right half plane, namely, $g(1) = 0$ with multiplicity n.

Furthermore, on $\{|z-1| = 1\}$, we have that

$$|f(z) - g(z)| = |\alpha| < 1 = |z-1| \leq |g(z)|$$

and so f and g have the same number of roots inside the circle $\{|z-1| < 1\}$ which is n.

Now, if f has any non-simple roots (any repeated roots), then f and f' would have a root in common.

However,

$$f'(z) = e^z(z-1)^n + ne^z(z-1)^{n-1} = e^z(z-1)^{n-1}[z-1+n]$$

and so if f and f' are simultaneously zero, then

$$\begin{aligned} f(z) &= 0 \\ e^z(z-1)^n &= a \\ e^z(z-1)^{n-1} &= \frac{a}{z-1} \\ f'(z) &= 0 \\ e^z(z-1)^{n-1}[z-1+n] &= 0 \\ \frac{a}{z-1}[z-1+n] &= 0 \end{aligned}$$

since $z = 1$ is not a root of f since $a \neq 0$ and $z - 1 = -n$ is not in $\{|z-1| < 1\}$ for any n, f' and f share no roots in $\{|z-1| < 1\}$. And since we have already shown that all of the roots of f in the right half plane lie in this circle, all n roots of f in the right-half plane have multiplicity 1 and so are simple. ✌

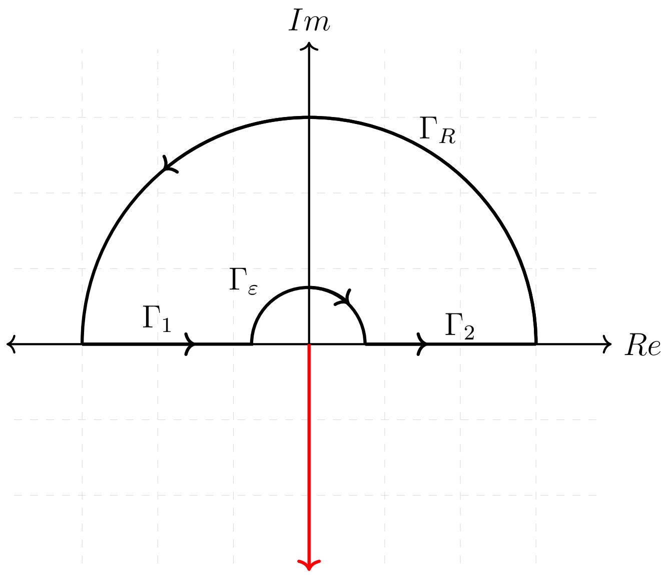
Problem 3. Evaluate

$$\int_0^\infty \frac{\log^2 x}{1+x^2} dx$$

Solution. This question is identical to **Fall 2013: Problem 1**. The same proof given there is provided here.

We will use "Ol' Faithful" the contour around the upper half plane avoiding the origin since every branch cut of $\log x$ intersects 0.

Then we take any branch which does not intersect the upper half plane (including the real line).



Let

$$I_1 = \int_{\Gamma_1} \frac{\log^2 z}{z^2+1} dz$$
$$I_2 = \int_{\Gamma_2} \frac{\log z}{z^2+1} dz$$
$$I_\varepsilon = \int_{\Gamma_\varepsilon} \frac{\log z}{z^2+1} dz$$
$$I_R = \int_{\Gamma_R} \frac{\log z}{z^2+1} dz$$

Note that

$$\begin{aligned}
I_1 &= \int_{-R}^{-\varepsilon} \frac{\log^2 x}{1+x^2}dx \\
&= \int_{R}^{\varepsilon} \frac{-(\log x + \pi i)^2}{1+x^2}dx \\
&= \int_{\varepsilon}^{R} \frac{\log^2 x + 2\pi i \log x - \pi^2}{1+x^2}dx \\
&= I_2 + 2\pi i \int_{\varepsilon}^{R} \frac{\log x}{1+x^2}dx - \pi^2 \int_{\varepsilon}^{R} \frac{1}{1+x^2}dx \\
&= I_2 - \pi^2(\tan^{-1}(R) - \tan^{-1}(\varepsilon)) + 2\pi i \int_{\varepsilon}^{R} \frac{\log x}{1+x^2}dx
\end{aligned}$$

Now,

$$\begin{aligned}
|I_R| &= \left| \int_{\Gamma_R} \frac{\log^2 z}{1+z^2}dz \right| \\
&\leq \int_0^{\pi} \frac{R|\log R + i\theta|^2}{R^2-1}d\theta \\
&\leq \pi \frac{R\log^2 R + 2R\pi \log R + R\pi^2}{R^2-1} \to 0 \qquad R \to \infty
\end{aligned}$$

since

$$\lim_{R\to\infty} \frac{\log^2 R}{R} = \lim_{R\to\infty} \frac{2\log R}{R} = \lim_{R\to\infty} \frac{2}{R} = 0$$

by L'Hopital's Rule and similarly, $\frac{\log R}{R} \to 0$.

Similarly,

$$\begin{aligned}
|I_\varepsilon| &\leq \int_{\pi}^{0} \frac{\varepsilon|\log \varepsilon + i\theta|^2}{\varepsilon^2-1}d\theta \\
&\leq \pi \frac{\varepsilon\log^2 \varepsilon + 2\varepsilon\pi \log \varepsilon + \varepsilon\pi^2}{\varepsilon^2-1} \to 0 \qquad \varepsilon \to 0
\end{aligned}$$

since

$$\lim_{\varepsilon\to 0} \varepsilon \log^2 \varepsilon = \lim_{\varepsilon\to 0} \frac{2\log\varepsilon}{\frac{-1}{\varepsilon}} = \lim_{\varepsilon\to 0} \frac{2}{\frac{1}{\varepsilon}} = 0$$

by L'Hopital's Rule.

Thus, by the Residue Theorem,

$$\begin{aligned}
2\pi i \text{Res}_{z=i} \frac{\log^2 z}{z^2+1} &= 2\pi i \frac{\log^2(i)}{i+i} \\
&= \pi \left(i\frac{\pi}{2} \right)^2 \\
&= -\frac{\pi^3}{4} \\
&= \lim_{R\to\infty} \lim_{\varepsilon\to 0} (I_1 + I_2 + I_\varepsilon + I_R \\
&= 2\int_0^\infty \frac{\log^2 x}{1+x^2} dx - \frac{\pi^3}{2} \\
\implies \int_0^\infty \frac{\log^2 x}{1+x^2} dx &= \frac{\pi^3}{8}
\end{aligned}$$

Note that since the residue is real this forces $\int_0^\infty \frac{\log x}{1+x^2} dx = 0.$ ✌

Problem 4. Denote $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Assume that $f : \mathbb{D} \to \mathbb{D}$ is analytic. Show that if $z_1 \neq z_2$ are fixed points of f in $\mathbb{D}$, then f is the identity map.

Solution. Since $\mathbb{D}$ is an open simply connected subset of $\mathbb{C}$ which is not all of $\mathbb{C}$, by the Riemann Mapping Theorem, there exists a $g : \mathbb{D} \to \mathbb{D}$ which is analytic, bijective, has an analytic inverse and is such that $g(z_1) = 0$. Say $g(z_2) = w$. Then

Then $g \circ f \circ g^{-1} : \mathbb{D} \to \mathbb{D}$ such that

$$g(f(g^{-1}(0))) = g(f(z_1)) = g(z_1) = 0.$$

Furthermore,

$$g(f(g^{-1}(w))) = g(f(z_2)) = g(z_2) = w$$

and so

$$|(g \circ f \circ g^{-1})(w)| = |w|$$

and so by Schwarz's Lemma, $g \circ f \circ g^{-1} = az$ where $|a| = 1$. However, since

$$g(f(g^{-1}(w))) = w = aw \implies a = 1.$$

Namely, $g \circ f \circ g^{-1}$ is the identity map and so f must be the identity map. ✌