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Complex Analysis Exam Fall 2016

Problem 1. Let $A = \{z \in \mathbb{C} : r < |z| < R\}$ for some $0 < r < R < \infty$. Prove that $f(z) = 1/z$ cannot be uniformly approximated in A by complex polynomials.

Solution. Let $\{p_n(z)\}$ be a sequence of polynomials converging uniformly to f on A.

Let $\varepsilon > 0$ and N such that $|f(z) - p_n(z)| < \varepsilon$ for all $n \geq N$ and all $z \in A$.

Then
$$|p_n(z)| - |f(z)| \leq |f(z) - p_n(z)| < \varepsilon$$
and so
$$|zp_n(z)| < \varepsilon|z| + 1.$$

Let $q_n(z) = zp_n(z)$. Then q_n is entire so on $\{|z| \leq M\}$
$$|q_n(z)| \leq \varepsilon M + 1.$$

Therefore, by the Cauchy Estimate, on $\{|z| \leq M\}$,
$$|q_n^{(2)}(z)| \leq \frac{2!(1 + \varepsilon M)}{M^2} \to 0 \qquad M \to \infty.$$

Therefore, there exists $a, b \in \mathbb{C}$ so $q_n(z) = az + b = zp_n(z)$ and so
$$p_n(z) = a + \frac{b}{z}$$
which is clearly a contradiction since p_n is a polynomial.

Thus, no such sequence can exist. ✌

Problem 2. Let $D = \mathbb{C}\backslash[-1,1]$. Prove that $f(z) = z^2 - 1$, for $z \in D$, has an analytic square root but does not have an analytic logarithm.

Solution. This question relies on two key facts

- f is an n^{th} root if and only if
$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz \quad \in n\mathbb{Z}$$
- f has an analytic logarithm if and only if f has an analytic n^{th} root for every n

Since these facts are not immediately obvious, they will be proved here.

Claim 1. Let f have no zeros in an open (*NOT* necessarily simply connected) region Ω, then f has an n^{th} root if and only if

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz \quad \in n\mathbb{Z}$$

for all closed curves $\gamma \subset \Omega$.

Proof. $\boxed{\implies}$ If f has an n^{th} root, then there exists an analytic $g(z)$ in Ω such that $f(z) = g^n(z)$.

Thus, for all $\gamma \subset \Omega$ closed,

$$\begin{aligned}\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz &= \frac{1}{2\pi i}\int_\gamma \frac{ng^{n-1}(z)g'(z)}{g^n(z)}dz\\ &= n\left[\frac{1}{2\pi i}\int_\gamma \frac{g'(z)}{g(z)}dz\right]\\ &= n(\text{number of zeros of } g \text{ in } \gamma)\\ &\in n\mathbb{Z}\end{aligned}$$

$\boxed{\impliedby}$ Let

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz \quad \in n\mathbb{Z}$$

for all closed curves $\gamma \subset \Omega$.

Now, fix some $z_0 \in \Omega$ and let γ_z be some curve in Ω from z_0 to some $z \in \Omega$.

Given γ_z and $\tilde{\gamma}_z$ two different paths, $\Gamma = \gamma_z \tilde{\gamma}_z^{-1}$ is a closed path and since

$$\frac{1}{n}\int_\Gamma \frac{f'(\xi)}{f(\xi)}d\xi \quad \in 2\pi i\mathbb{Z}$$

we have that $\frac{1}{n}\int_{\gamma_z} \frac{f'(\xi)}{f(\xi)}d\xi$ differs by choice of path by integer multiples of $2\pi i$.

Let

$$h(z) = e^{\frac{1}{n}\int_{\gamma_z} \frac{f'(\xi)}{f(\xi)}d\xi}.$$

Then h is independent of choice of γ_z and so is well defined.

Finally, since f has no zeros in Ω, then $\frac{f'}{f}$ is analytic in Ω and so $h'(z)$ exists and

$$h'(z) = e^{\frac{1}{n}\int_{\gamma_z} \frac{f'(\xi)}{f(\xi)}d\xi}\frac{f'(z)}{nf(z)} = h(z)\frac{f'(z)}{nf(z)}.$$

$$\begin{aligned}\frac{d}{dz}\frac{f(z)}{h^n(z)} &= \frac{f'(z)h^n(z) - nf(z)h^{n-1}(z)h'(z)}{h^{2n}(z)} \\ &= \frac{f'(z)h^n(z) - nf(z)h^{n-1}(z)h(z)\frac{f'(z)}{nf(z)}}{h^{2n}(z)} \\ &= \frac{f'(z)h^n(z) - h^n(z)f'(z)}{h^{2n}(z)} \\ &= 0\end{aligned}$$

Therefore, $\frac{f(z)}{h^n(z)} = c$ some constant and so $f(z) = ch^n(z) = (c^{1/n}h(z))^n$.

Finally, then f must have an n^{th} root. ✌

Now,

$$\frac{f'(z)}{f(z)} = \frac{2z}{z^2-1}$$

which has poles at $-1, 1 \notin D$. Therefore, any closed curve $\gamma \subset D$ either dodges the line segment $[-1, 1]$ and so cannot contain either of the two poles, or it wraps around the line segment $[-1, 1]$ in which case, it must contain both poles.

Namely,

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz \in 2\mathbb{Z}$$

for all closed curves $\gamma \subset D$.

Thus, by **Claim 1** a square root of f exists.

Finally, we note that f has an analytic logarithm if and only if f has an analytic n^{th} root for all n.

Claim 2. If f has no zeros in Ω, then f has an analytic logarithm if and only if f has an analytic n^{th} root for all n.

Proof. $\boxed{\Longrightarrow}$ If f has an analytic logarithm, then there exists an analytic $g(z)$ such that $f(z) = e^{g(z)}$.

Thus, for all closed curves γ,

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i}\int_\gamma \frac{e^{g(z)}g'(z)}{e^{g(z)}}dz = \frac{1}{2\pi i}\int_\gamma g'(z)dz = 0$$

since g' is also analytic.

Since $0 \in n\mathbb{Z}$ for all n, by **Claim 1**, f has an n^{th} root for all n.

$\boxed{\Longleftarrow}$ If f has an n^{th} root for all n, then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = 0$$

for all γ closed in the domain.

Namely,

$$g(z) = \frac{1}{n}\int_{z_0}^{z} \frac{f'(z)}{f(z)}dz$$

is an analytic function and well defined for all z and all fixed z_0.

Therefore, using the same idea as **Claim 1**, we get that $f = e^{g(z)}$ where g is analytic and so f has an analytic logarithm in Ω. ✌

Therefore, by **Claim 2**, f cannot have an analytic logarithm, since $\gamma = |z| = 2$ gives

$$\frac{1}{2\pi i}\int_{|z|=2} \frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i}\int_{|z|=2} \frac{2z}{z^2-1}dz = \text{Res}_{z=1}\frac{2z}{z^2-1} + \text{Res}_{z=-1}\frac{2z}{z^2-1} = \frac{2}{1+1} + \frac{-2}{-1-1} = 2 \neq 0.$$

Note that to actually determine the value of $\int_\gamma \frac{f'}{f}dz$, we can view the integral in $\mathbb{C}$ and apply the argument principle or residue theorem there.

Since the integral exists in D, the two values must be the same.

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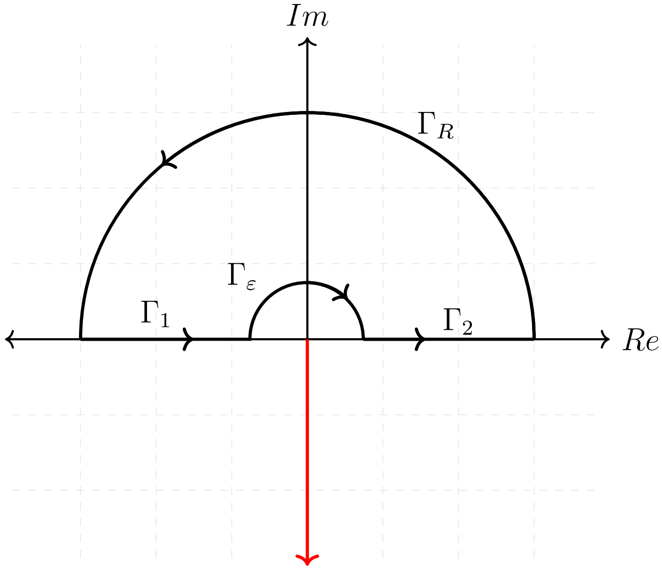
Problem 3. Evaluate

$$\int_0^\infty \frac{\log x}{1+x^2}dx$$

Solution. We will use "Ol' Faithful" the contour around the upper half plane avoiding the origin since every branch cut of $\log x$ intersects 0.

Then we take any branch which does not intersect the upper half plane (including the real line).



Let

$$I_1 = \int_{\Gamma_1} \frac{\log z}{z^2+1}dz$$
$$I_2 = \int_{\Gamma_2} \frac{\log z}{z^2+1}dz$$
$$I_\varepsilon = \int_{\Gamma_\varepsilon} \frac{\log z}{z^2+1}dz$$
$$I_R = \int_{\Gamma_R} \frac{\log z}{z^2+1}dz$$

Now,

$$\begin{aligned}
I_1 &= \int_{\Gamma_1} \frac{\log z}{z^2+1} dz \\
&= \int_{-R}^{-\varepsilon} \frac{\log x}{x^2+1} dx \\
&= \int_{R}^{\varepsilon} \frac{-\log(-x)}{x^2+1} dx \\
&= \int_{\varepsilon}^{R} \frac{\log x + \pi i}{x^2+1} dx \\
&= I_2 + \pi i \tan^{-1}(x)\Big|_{\varepsilon}^{R} \\
&= I_2 + \pi i(\tan^{-1}(R) - \tan^{-1}(\varepsilon))
\end{aligned}$$

$$\begin{aligned}
|I_R| &= \left| \int_{\Gamma_R} \frac{\log z}{z^2+1} dz \right| \\
&\leq \int_{\Gamma_R} \frac{|\log z|}{|z^2+1|} d|z| \\
&\leq \int_{\Gamma_R} \frac{|\log z|}{|z|^2-1} d|z| \\
&= \int_0^{\pi} \frac{R|\log(Re^{i\theta})|}{R^2-1} d\theta \\
&= \int_0^{\pi} \frac{R|\log(R)+i\theta|}{R^2-1} d\theta \\
&\leq \int_0^{\pi} \frac{R\log(R)+R\pi}{R^2-1} d\theta \\
&= \pi \frac{R(\log R + \pi)}{R^2-1} \to 0 \qquad R \to \infty
\end{aligned}$$

since

$$\lim_{R\to\infty} \frac{R\log R}{R^2-1} = \lim_{R\to\infty} \frac{\log R + 1}{2R} \lim_{R\to\infty} \frac{1}{R} = 0$$

by L'Hopital's Rule.

Similarly,

$$\begin{aligned}|I_\varepsilon| &= \left|\int_{\Gamma_\varepsilon} \frac{\log z}{z^2+1} dz\right| \\ &\leq \int_{\Gamma_\varepsilon} \frac{|\log z|}{1-|z|^2} d|z| \\ &= \int_0^\pi \frac{\varepsilon|\log(\varepsilon e^{i\theta})|}{\varepsilon^2-1} d\theta \\ &\leq \int_0^\pi \frac{\varepsilon\log(\varepsilon)+\varepsilon\pi}{\varepsilon^2-1} d\theta \\ &= \pi\frac{\varepsilon(\log\varepsilon+\pi)}{\varepsilon^2-1} \to 0 \qquad \varepsilon\to 0\end{aligned}$$

since

$$\lim_{\varepsilon\to 0}\varepsilon\log\varepsilon = \lim_{\varepsilon\to 0}\frac{\log\varepsilon}{\frac{1}{\varepsilon}} = \lim_{\varepsilon\to 0}\frac{\frac{1}{\varepsilon}}{-\frac{1}{\varepsilon^2}} = \lim_{\varepsilon\to 0} -\varepsilon = 0$$

by L'Hopital's Rule.

Finally, by the Residue Theorem,

$$\begin{aligned}2\pi i \mathrm{Res}_{z=i}\frac{\log z}{z^2+1} &= 2\pi i\frac{\log(i)}{i+i} \\ &= \pi\left(\log|i| + i\frac{\pi}{2}\right) \\ &= i\frac{\pi^2}{2} \\ &= \lim_{R\to\infty}\lim_{\varepsilon\to 0}(I_1+I_2+I_R+I_\varepsilon) \\ &= 2\int_0^\infty \frac{\log x}{1+x^2}dx + \pi i\left(\frac{\pi}{2}\right) \\ \implies \int_0^\infty \frac{\log x}{1+x^2}dx = 0\end{aligned}$$

Note that this is consistent with what we found in **Fall 2013: Problem 1**. ✌

Problem 4. Show that the range of a nonconstant entire function is dense in $\mathbb{C}$.

Solution. Let f be nonconstant and entire. Assume that $f(\mathbb{C})$ is not dense in $\mathbb{C}$.

Then there exists some $z_0 \in \mathbb{C}$ and some $\rho > 0$ such that $f(\mathbb{C}) \cap B_\rho(z_0) = \varnothing$.

Namely,

$$|f(z) - z_0| \geq \rho$$

for all $z \in \mathbb{C}$.

However, then we can let $g(z) = \frac{1}{f(z)-z_0}$. Then g is clearly entire but

$$|g(z)| = \frac{1}{|f(z) - z_0|} \leq \frac{1}{\rho}$$

and so g is bounded on $\mathbb{C}$, so by Louiville's g is constant.

However, then f is constant which is a contradiction.

Thus, if f is non-constant then it is dense in $\mathbb{C}$.

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