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Complex Analysis Exam Spring 2015

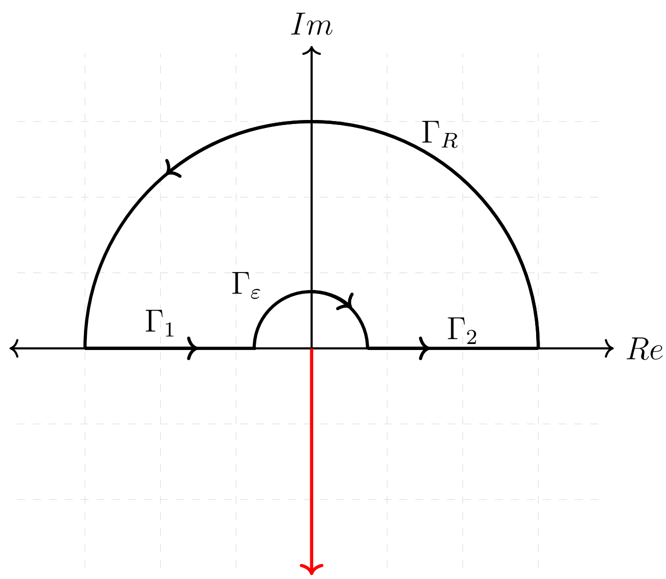
Problem 1. Evaluate the integral

$$\int_0^\infty \frac{x^{1/3}}{1+x^2}dx$$

being careful to justify your answer.

Solution. We will use "Ol' Faithful" the contour around the upper half plane avoiding the origin since every branch cut of $x^{1/3} = e^{\frac{1}{3}\log x}$ intersects 0.

Then we take any branch which does not intersect the upper half plane (including the real line).



Let

$$I_1 = \int_{\Gamma_1} \frac{e^{1/3 \log z}}{z^2+1} dz$$
$$I_2 = \int_{\Gamma_2} \frac{e^{1/3 \log z}}{z^2+1} dz$$
$$I_\varepsilon = \int_{\Gamma_\varepsilon} \frac{e^{1/3 \log z}}{z^2+1} dz$$
$$I_R = \int_{\Gamma_R} \frac{e^{1/3 \log z}}{z^2+1} dz$$

Note that

$$\begin{aligned} I_1 &= \int_{\Gamma_1} \frac{e^{1/3 \log z}}{z^2+1} dz \\ &= \int_{-R}^{-\varepsilon} \frac{e^{1/3 \log x}}{x^2+1} dx \\ &= \int_{R}^{\varepsilon} \frac{-e^{1/3 \log(-x)}}{x^2+1} dx \\ &= \int_{\varepsilon}^{R} \frac{e^{1/3(\log x + i\pi}}{x^2+1} dx \\ &= e^{i\pi/3} \int_{\varepsilon}^{R} \frac{x^{1/3}}{x^2+1} dx \\ &= e^{i\pi/3} I_2 \end{aligned}$$

Now

$$\begin{aligned} |I_R| &= \left| \int_{\Gamma_R} \frac{e^{1/3 \log z}}{z^2+1} dz \right| \\ &\leq \int_{\Gamma_R} \frac{|e^{1/3 \log z}|}{|z^2+1|} d|z| \\ &\leq \int_0^\pi \frac{R e^{1/3 \log R}}{R^2-1} d\theta \qquad z = Re^{i\theta} \\ &= \frac{\pi R^{4/3}}{R^2-1} \to 0 \qquad R \to \infty \end{aligned}$$

and

$$\begin{aligned} |I_\varepsilon| &= \left| \int_{\Gamma_\varepsilon} \frac{e^{1/3 \log z}}{z^2+1} dz \right| \\ &\leq \int_{\Gamma_\varepsilon} \frac{|e^{1/3 \log z}|}{|z^2+1|} d|z| \\ &\leq \int_\pi^0 \frac{\varepsilon e^{1/3 \log \varepsilon}}{1-\varepsilon^2} d\theta \qquad z = \varepsilon e^{i\theta} \\ &= \frac{\pi \varepsilon^{4/3}}{1-\varepsilon^2} \to 0 \qquad \varepsilon \to 0. \end{aligned}$$

Thus, by the Residue Theorem,

$$\begin{aligned}
2\pi i \text{Res}_{z=i} \frac{e^{1/3 \log z}}{z^2+1} &= 2\pi i \frac{e^{1/3 \log(i)}}{i+i} \\
&= \pi e^{1/3(i\pi/2)} \\
&= \pi e^{i\pi/6} \\
&= \lim_{R\to\infty} \lim_{\varepsilon \to 0} (I_1 + I_2 + I_R + I_\varepsilon) \\
&= (e^{i\pi/3}+1) \int_0^\infty \frac{x^{1/3}}{x^2+1} dx \\
\implies \int_0^\infty \frac{x^{1/3}}{x^2+1} dx &= \frac{\pi e^{i\pi/6}}{e^{i\pi/3}+1} \\
&= \frac{\pi}{e^{i\pi/6}+e^{i\pi/6}} \\
&= \frac{\pi/2}{\cos(\pi/6)} \\
&= \frac{\pi/2}{\sqrt{3}/2} \\
&= \frac{\pi}{\sqrt{3}}
\end{aligned}$$

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Problem 2. Let

$$f(z) = \sum_{n=0}^{\infty} z^{n!}.$$

(a) Show that f is analytic in the open unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$.

(b) Show that f can not be analytically continued to any open set properly containing D. (Hint: First consider $z = r^{2\pi ip}/q$ where p and q are integers.

Solution.

(a) f is holormophic if and only if the sum converges uniformly. For $|z| < r < 1$, we have that $|z|^{n!} \leq r^{n!} \leq r^n$ since $r < 1$, and so f converges uniformly by Weierstrass M-test on $\{|z| < r\}$. Since this holds for all $r < 1$, we have that f converges uniformly and is therefore analytic on D.

(b) Let $D \subsetneq U$ open. Since $U \neq D$ and U is open, U must contain some z_0 with $|z_0| = 1$.

Again, since U is open, any neighborhood of z_0 must contain some $z = e^{i\pi/m}$ for $m \in \mathbb{Z}$ since $|z| = 1$.

However, then

$$f(z) = \sum_{n=0}^{m-1} e^{in!\pi/m} + \sum_{n=m}^{\infty} e^{i\pi n!/m} = \sum_{n=0}^{m-1} e^{in!\pi/m} + \sum_{n=m}^{\infty} 1 = \infty$$

since $n!/m \in \mathbb{Z}$ for $n \geq m$.

Now, f has a non-removable singularity in every neighborhood of z_0.

Namely, f cannot be extended to a neighborhood to z_0 since it will not converge in any punctured neighborhood of z_0.

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Problem 3. Let A be an open subset of $\mathbb{C}$, and suppose $u(x,y)$ is twice continuously differentiable harmonic function on A.

(a) Show that if A is simply connected, then there exists an analytic function f on A such that $u = \Re\mathrm{e}(f)$. Hint: First find g so that $\partial u/\partial x = \Re\mathrm{e}(g)$).

(b) Find f explicitly when $A = \mathbb{C}$ and $u(x,y) = e^x \cos(y) + xy$.

(c) Give an example in which A is not simply connected and f as in (a) does not exist.

Solution.

(a) Let $v(x,y) = \int_{y_0}^{y} u_x(x,t)dt + g(x) - g(x_0)$. Then in order for Cauchy Riemann to be satisfied,

$$\begin{aligned}
-u_y(x,y) &= v_x(x,y) \\
&= \int_{y_0}^{y} u_{xx}(x,t)dt + g'(x) \\
&= \int_{y_0}^{y} -u_{yy}(x,t)dt + g'(x) \qquad (1)\\
&= -u_y(x,y) + u_y(x,y_0) + g'(x) \\
\implies -u_y(x,y_0) &= g'(x) \\
\implies \int_{x_0}^{x} -u_y(s,y_0)dx &= g(x) - g(x_0)
\end{aligned}$$

with (1) since u is harmonic so $u_{xx} + u_{yy} = 0$.

Therefore,

$$v(x,y) = \int_{y_0}^{y} u_x(x,t)dt - \int_{x_0}^{x} u_y(s,y_0)dx.$$

Then

$$\begin{aligned}
v_x(x,y) &= \int_{y_0}^{y} u_{xx}(x,t)dt - u_y(x,y_0) \\
&= \int_{y_0}^{y} -u_{yy}(x,t)dt - u_y(x,y_0) \\
&= -u_y(x,y) + u_y(x,y_0) - u_y(x,y_0) \\
&= -u_y(x,y)
\end{aligned}$$

and $v_y(x,y) = u_x(x,y)$ since $u_y(x,y_0)$ is constant in y.

Therefore, $f = u + iv$ is analytic by the Cauchy Riemann with $\Re\mathrm{e}(f) = u$.

(b) Let $u(x, y) = e^x \cos(y) + xy$. Then

$$u_y = -e^x \sin(y) + x$$

and so

$$\begin{aligned} v(x,y) &= -\int u_y(x,y)dx \\ &= -\int -e^x \sin(y) + xdx \\ &= -\left(-e^x \sin(y) + \frac{x^2}{2} + h(y)\right) \\ &= e^x \sin(y) - \frac{x^2}{2} + h(y) \end{aligned}$$

Since

$$u_x = e^x \cos(y) + y = v_y = e^x \cos(y) + h'(y) \implies h'(y) = y$$

and so $h = \frac{y^2}{2}$ so

$$v(x,y) = e^x \sin(y) - \frac{x^2}{2} + \frac{y^2}{2}$$

and

$$f(x,y) = e^x \sin(y) + xy + i(e^x \sin(y) - \frac{x^2}{2} + \frac{y^2}{2}).$$

Once can verify that

$$u_y = -e^x \sin(y) + x = -(e^x \sin(y) + x) = -v_x$$

and so indeed f is analytic by Cauchy-Riemann.

(c) Let $A = \mathbb{C}\backslash\{0\}$. Then A is not simply connected. Let $u(x, y) = \frac{1}{2}\log(x^2 + y^2)$. Then

$$u_x = \frac{2x}{2(x^2 + y^2)} = \frac{x}{x^2 + y^2}$$

and

$$u_y = \frac{y}{x^2 + y^2}$$

so

$$\begin{aligned} u_{xx} &= \frac{x^2 + y^2 - 2x^2}{(x^2 + y^2)^2} \\ &= \frac{y^2 - x^2}{(x^2 + y^2)^2} \\ &= -\frac{x^2 - y^2}{(x^2 + y^2)^2} \\ &= -u_{yy} \end{aligned}$$

so u is harmonic in A.

However, u cannot be the real part of an analytic function, since it is the real part of the complex $\log|z| + iarg(z)$ which is not well defined on A. Namely, there is not analytic continuation of $\log z$ (any branch cut) to A and so u cannot be the real part of an analytic function.

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Problem 4. Determine whether it is possible for a function f to be analytic in a neighborhood of 0 and take the values $\frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{6}, \frac{1}{6}, \frac{1}{8}, \frac{1}{8}, ...$ at the points $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, ...$.

Solution. Assume such an f does exist. By continuity,

$$\lim_{n\to\infty} f\left(\frac{1}{n}\right) = f(0) = 0.$$

Therefore,

$$\begin{aligned}\lim_{h\to 0}\frac{f(h)-f(0)}{h} &= \lim_{n\to\infty}\frac{f\left(\frac{1}{n}\right)}{1/n}\\ &= \lim_{n\to\infty}\begin{cases}\frac{1/n}{1/n} & n \text{ even}\\ \frac{1/(n+1)}{1/n} & n \text{ odd}\end{cases}\\ &= 1\end{aligned}$$

so f is injective in a neighborhood of 0, which is clearly not true.

Thus, f cannot exist. ✌