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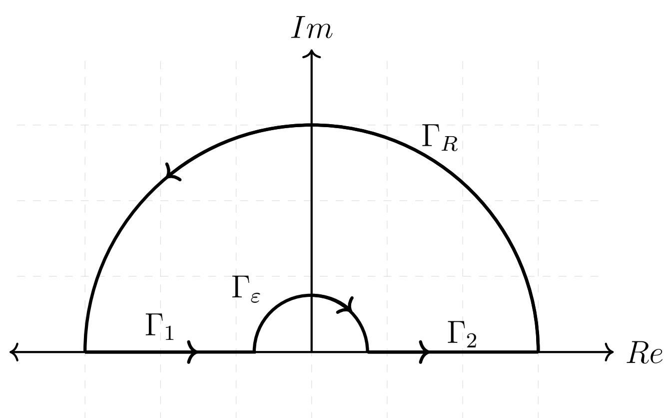
Complex Analysis Exam Fall 2015

Problem 1. Evaluate the integral

$$\int_0^\infty \frac{\sin^2 x}{x^2} dx$$

being careful to justify your answer.

Solution. We will use "Ol' Faithful" the contour around the upper half plane avoiding the origin.



Let

$$I_1 = \int_{\Gamma_1} \frac{1-e^{2iz}}{z^2} dz$$
$$I_2 = \int_{\Gamma_2} \frac{1-e^{2iz}}{z^2} dz$$
$$I_\varepsilon = \int_{\Gamma_\varepsilon} \frac{1-e^{2iz}}{z^2} dz$$
$$I_R = \int_{\Gamma_R} \frac{1-e^{2iz}}{z^2} dz$$

Then note that

$$\begin{aligned} I_1 &= \int_{\Gamma_1} \frac{1-e^{2iz}}{z^2} dz \\ &= \int_{-R}^{-\varepsilon} \frac{1-e^{2ix}}{x^2} dx \\ \int_R^{\varepsilon} &- \frac{1-e^{-2ix}}{x^2} dx \\ &= \int_{\varepsilon}^{R} \frac{1-e^{-2ix}}{x^2} dx \end{aligned}$$

Next, note that

$$I_1+I_2 = \int_{\varepsilon}^{R} \frac{2-e^{2ix}-e^{-2ix}}{x^2} dx = \int_{\varepsilon}^{R} \frac{2\left(1-\frac{e^{2ix}+e^{-2ix}}{2}\right)}{x^2} dx = 2\int_{\varepsilon}^{R} \frac{1-\cos(2x)}{x^2} = 4\int_{\varepsilon}^{R} \frac{\sin^2 x}{x^2} dx$$

Thus,

$$\begin{aligned} |I_R| &= \left| \int_{\Gamma_R} \frac{1-e^{2iz}}{z^2} dz \right| \\ &\leq \int_{\Gamma_R} \frac{|1-e^{2iz}|^2}{|z|^2} dz \\ &\leq \int_0^{\pi} \frac{1+|e^{2iRe^{i\theta}}|}{R} d\theta \qquad z = Re^{i\theta} \\ &= \int_0^{\pi} \frac{1+e^{-2R\sin\theta}}{R} d\theta \qquad 0 \leq \sin\theta \leq 1 \implies e^{-2R\sin\theta} \leq 1 \\ &\leq \int_0^{\pi} \frac{2}{R} d\theta \\ &= \frac{2\pi}{R} \to 0 \qquad R \to \infty \end{aligned}$$

Now, for I_ε, note that $\frac{1-e^{2iz}}{z^2}$ has an isolated pole of order 2 at 0. Thus, we can write

$$\frac{1-e^{2iz}}{z^2} = \frac{a}{z^2} + \frac{b}{z} + f(z)$$

with f analytic at 0,

$$b = \text{Res}_{z=0} \frac{1-e^{2iz}}{z^2} = \frac{d}{dz}(1-e^{2iz})\Big|_0 = -2ie^{2iz}\Big|_0 = -2i$$

and

$$a = \lim_{z\to 0} z^2 \frac{1-e^{2iz}}{z^2} = \lim_{z\to 0}(1-e^{2iz}) = 1-1 = 0.$$

Thus, for ε small enough,

$$\begin{aligned} I_\varepsilon &= \int_{\Gamma_\varepsilon} \frac{1-e^{2iz}}{z^2} dz \\ &= \int_{\Gamma_\varepsilon} \frac{-2i}{z} + f(z)dz \\ &= \int_\pi^0 2 + i\varepsilon e^{i\theta} f(\varepsilon e^{i\theta}) d\theta \qquad z = \varepsilon e^{i\theta} \\ &= -2\pi + \int_\pi^0 i\varepsilon e^{i\theta} f(\varepsilon e^{i\theta}) d\theta \to 2\pi i \qquad \varepsilon \to 0 \end{aligned}$$

since f is analytic so

$$\lim_{\varepsilon\to 0} \int_\pi^0 i\varepsilon e^{i\theta} f(\varepsilon e^{i\theta}) d\theta = \int_\pi^0 \lim_{\varepsilon \to 0} i\varepsilon e^{i\theta} f(\varepsilon e^{i\theta}) d\theta = 0.$$

Finally, by the residue theorem,

$$\begin{aligned} 0 &= \lim_{R\to\infty} \lim_{\varepsilon\to 0} (I_1 + I_2 + I_R + I_\varepsilon) \\ &= 4\int_0^\infty \frac{\sin^2 x}{x^2} dx - 2\pi \\ \implies \int_0^\infty \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2} \end{aligned}$$

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Problem 2. Determine the number of roots of $f(z) = z^9 + z^6 + z^5 + 8z^3 + 1$ inside the annulus $1 < |z| < 2$.

Solution. Let $g(z) = z^9$. Now, on {|z|=2} we get that

$$\begin{aligned}|f(z) - g(z)| &= |z^6 + z^5 + 8z^3 + 1|\\ &\leq 2^6 + 2^5 + 8 \cdot 2^3 + 1\\ &= 161\\ &< 512\\ &= |z|^9\\ &= |g(z)|\end{aligned}$$

and so by Rouche's Theorem, g and f have the same number of zeros inside $\{|z| < 2\}$ which is 9.

Let $g(z) = 8z^3$, then on $\{|z| = 1\}$

$$\begin{aligned}|f(z) - g(z)| &= |z^9 + z^6 + z^5 + 1|\\ &\leq 1 + 1 + 1 + 1\\ &= 4\\ &< 8\\ &= 8|z|^3\\ &= |g(z)|\end{aligned}$$

and so by Rouche's, f and g have the same number of zeros inside $\{|z| < 1\}$ which is 3. Therefore, f has $9 - 3 = 6$ zeros inside the annulus $\{1 < |z| < 2\}$. ✌

Problem 3. Suppose that f is holomorphic on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and suppose that for $z \in \mathbb{D}$ one has $\Re(f(z)) > 0$ and $f(0) = 1$. Prove that $|f(z)| \leq \frac{1+|z|}{1-|z|}$ for all $z \in \mathbb{D}$.

Solution. Since f sends the unit disk to the right half plane, if sends the unit disk to the upper half plane.

Thus, let $T(z) = \frac{z-i}{z+i}$. Then T sends the upper half plane to the unit disk.

Furthermore,

$$T(if(0)) = T(i) = 0$$

and so $T(if(z))$ is a map from the disk to the disk preserving the origin.

Therefore, by Schwarz' Lemma,

$$\begin{aligned}
|T(if(z))| &\leq |z| \\
\left|\frac{if(z)-i}{if(z)+i}\right| &\leq |z| \\
\frac{|f(z)-1|}{|f(z)+1|} &\leq |z| \\
|f(z)-1| &\leq |z||f(z)+1| \\
|f(z)|-1 \leq |f(z)-1| &\leq |z||f(z)+1| \leq |z|(|f(z)|+1) \\
(1-|z|)|f(z)| &\leq 1+|z| \\
|f(z)| &\leq \frac{1+|z|}{1-|z|}
\end{aligned}$$

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Problem 4. For $a_n = 1 - \frac{1}{n^2}$, let

$$f(z) = \prod_{n=1}^{\infty} \frac{a_n - z}{1 - a_n z}.$$

(a) Show that f defines a holomorphic function on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

(b) Prove that f does not have an analytic continuation to any larger disk $\{z \in \mathbb{C} : |z| < r\}$ for some $r > 1$.

Solution.

(a) Since $|a_n| < 1$ for all n, f defines an infinite product of analytic functions in the disk. Note that $T_n(z) = \frac{a_n - z}{1 - a_n z}$ is actually an automorphism of the disk.

Thus, f is analytic in the disk if $\sum_{n=1}^{\infty}(T_n(z) - 1)$ converges absolutely and uniformly.

$$\begin{aligned}
T_n(z) - 1 &= \frac{a_n - z}{1 - a_n z} - 1 \\
&= \frac{1 - \frac{1}{n^2} - z}{1 - (1 - \frac{1}{n^2})z} - 1 \\
&= \frac{n^2 - 1 - n^2 z}{n^2 - (n^2 - 1)z} - 1 \\
&= \frac{n^2 - 1 - n^2 z}{n^2 - n^2 z + z} - 1 \\
&= \frac{n^2 - 1 - n^2 z - (n^2 - n^2 z + z)}{n^2 - n^2 z + z} \\
&= \frac{-z - 1}{n^2 - n^2 z - z} \\
&= \frac{z + 1}{n^2 z + z - n^2}
\end{aligned}$$

Now, for all $|z| < r < 1$, we have that

$$|T_n(z) - 1| \leq \frac{|z| + 1}{|z - 1|n^2 - |z|} < \frac{2}{(1 - r)n^2 - 1}$$

which converges uniformly as a series. Since r was arbitrary, we have that the sum converges uniformly in the unit disk.

Thus, f defines an analytic function.

(b) If f has an analytic continuation at some larger disk, then f must have an analytic continuation at 1, since any larger disk will contain 1.

Note that $f(a_n) = 0$ for all n, and $a_n \to 1$. Namely, if f has an analytic continuation g on a larger disk then $g(a_n) = 0$ for all n and since there is an accumulation point in any larger disk, by the identity theorem $g \equiv 0$.

This is clearly a contradiction so g cannot exist.

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