Kayla Orlinsky

Complex Analysis Exam Spring 2014

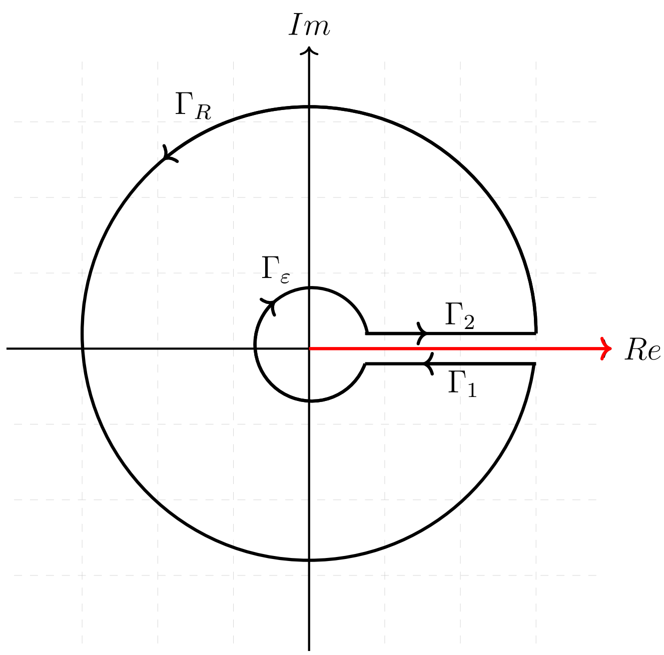
Problem 1. For $a > 0$, evaluate the integral

$$\int_0^\infty \frac{\log x}{(a+x)^3},$$

being careful to justify your methods.

Solution. Note that since $x = -a$ on the negative real axis is pole of the function. Namely, "Ol Faithful" will intersect a pole in this case.

Thus, we will use the dreaded "pac man" contour around the whole plane with the branch cut on the real axis.



Now, we examine $\frac{\log^2 z}{(a+z)^3}$ simply because **Fall 2013: Problem 1** showed that integrating $\log^2 z$ inadvertently gave us the value of $\log z$.

Let

$$
\begin{aligned}
I_1 &= \int_{\Gamma_1} \frac{\log^2 z}{(a+z)^3} dz \\
I_2 &= \int_{\Gamma_2} \frac{\log^2 z}{(a+z)^3} dz \\
I_\varepsilon &= \int_{\Gamma_\varepsilon} \frac{\log^2 z}{(a+z)^3} dz \\
I_R &= \int_{\Gamma_R} \frac{\log^2 z}{(a+z)^3} dz
\end{aligned}
$$

Note that

$$
\begin{aligned}
I_1 &= \int_{\Gamma_1} \frac{\log^2 z}{(a+z)^3} dz \\
&= \int_R^\varepsilon \frac{(\log x + i2\pi)^2}{(a+x)^3} dx \\
&= \int_R^\varepsilon \frac{\log^2 x + i4\pi \log x - 4\pi^2}{(a+x)^3} dx \\
&= -I_2 - i4\pi \int_\varepsilon^R \frac{\log x}{(a+x)^3} dx + 4\pi^2 \int_\varepsilon^R \frac{1}{(a+x)^3} dx \\
&= -I_2 - i4\pi \int_\varepsilon^R \frac{\log x}{(a+x)^3} dx - 4\pi^2 \frac{1}{2(a+x)^2}\Big|_\varepsilon^R \\
&= -I_2 - i4\pi \int_\varepsilon^R \frac{\log x}{(a+x)^3} dx - 2\pi^2 \left[\frac{1}{(a+R)^2} - \frac{1}{(a+\varepsilon)^2}\right]
\end{aligned}
$$

Now,

$$
\begin{aligned}
|I_R| &= \left| \int_{\Gamma_R} \frac{\log^2 z}{(a+z)^3} dz \right| \\
&\leq \int_{\Gamma_R} \frac{|\log^2 z|}{|a+z|^3} d|z| \\
&\leq \int_0^{2\pi} \frac{R|\log R + i\theta|^2}{R^3 - a^3} d\theta \\
&\leq 2\pi \frac{R \log^2 R}{R^3 - a^3} + 2\pi \frac{2\pi R}{R^3 - a^3} \to 0 \qquad R \to \infty
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
|I_\varepsilon| &\leq \int_0^{2\pi} \frac{\varepsilon |\log \varepsilon + i\theta|^2}{a^3 - \varepsilon^3} d\theta \\
&\leq 2\pi \frac{\varepsilon \log^2 \varepsilon}{a^3 - \varepsilon^3} + 2\pi \frac{2\pi\varepsilon}{a^3 - \varepsilon^3} \to 0 \qquad \varepsilon \to 0
\end{aligned}
$$

Thus, by the Residue Theorem,

$$\begin{aligned}
2\pi i \text{Res}_{z=-a} \frac{\log^2 z}{(a+z)^3} &= 2\pi i \frac{1}{2!} \frac{d^2}{dz^2} \log^2 z \Big|_{-a} \\
&= \pi i \frac{d}{dz} \frac{2\log z}{z} \Big|_{-a} \\
&= 2\pi i \left[\frac{1-\log z}{z^2} \right] \Big|_{-a} \\
&= 2\pi i \left[\frac{1-\log(-a)}{a^2} \right] \\
&= 2\pi i \left[\frac{1-\log a - \pi i}{a^2} \right] \\
&= \lim_{R\to\infty} \lim_{\varepsilon \to 0} (I_1 + I_2 + I_R + I_\varepsilon) \\
&= -i4\pi \int_0^\infty \frac{\log x}{(a+x)^3} dx + \frac{2\pi^2}{a^2} \\
\implies \int_0^\infty \frac{\log x}{(a+x)^3} dx &= \frac{-1}{i2\pi} \left[2\pi i \left[\frac{1-\log a - \pi i}{a^2} \right] - \frac{2\pi^2}{a^2} \right] \\
&= \frac{\log a - 1 + \pi i}{2a^2} - \frac{i\pi}{2a^2} \\
&= \frac{\log a - 1}{2a^2}
\end{aligned}$$

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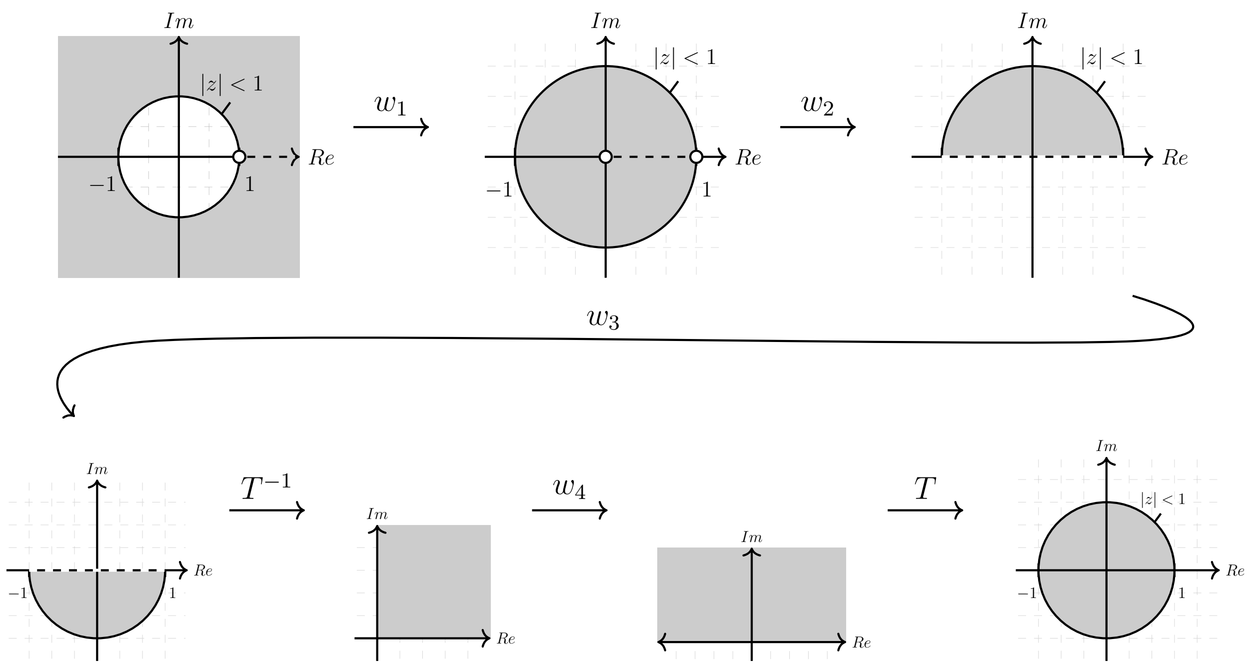
Problem 2. Find a conformal mapping of the region $\{z : |z| > 1\} \backslash (1, \infty)$ onto the open unit disk $\{z : |z| < 1\}$. You may give your answer as the composition of several mappings, so long as each mapping is precisely described.

Solution. Let

$$
\begin{aligned}
T(z) &= \frac{z-i}{z+i} \\
w_1(z) &= \frac{1}{z} \\
w_2(z) &= z^{\frac{1}{2}} \qquad \text{branch at } [0, \infty) \\
w_3(z) &= -z \\
w_4(z) &= z^2
\end{aligned}
$$



Problem 3. Suppose that f_n are analytic functions on a connected open set $U \subset \mathbb{C}$ and that $f_n \to f$ uniformly on compact subsets of U. In each case indicate the main setps in the proofs of the following stnadard results.

(a) f is analytic in U;

(b) $f_n' \to f'$ uniformly on compact subsets of U;

(c) if $f_n(z) \neq 0$ for all n and all $z \in U$, then either $f(z) \neq 0$ for all $z \in U$ or else $f \equiv 0$.

Solution. ***What a poorly worded question... what are "main steps"?***

(a) Let $\varepsilon > 0$. Then, there exists an N such that $|f_n(z) - f(z)| < \varepsilon$ for all $n \geq N$ and all $z \in K$ compact $\subset U$.

Clearly this forces any singularities of f to be isolated. Else uniform convergene via analytic functions is not possible.

Furthermore, if f has a removable singularity at $z_0 \in K$, then $f(z_0) = w$ but $\lim_{z\to z_0} f(z) = w' = \lim_{n\to\infty} f_n(z_0)$, so uniform convergence is not possible unless $w' = w$, namely f cannot have a removable singularity.

Similarly, if f has a pole or essential singularity in K, then again this will contradict uniform convergence. So f is analytic in all compact subsets of U. Namely, f is analytic in U.

(b) Let $K \subset U$ be compact. Let $z \in K$ and ρ such that $B_\rho(z) \subset K$. Then let $\varepsilon > 0$ and N such that $|f_n(z) - f(z)| < \varepsilon$ for all $n \geq N$ and all $z \in K$. Then

$$\begin{aligned}
|f_n'(z) - f'(z)| &= \left| \int_{|\xi - z| = \rho} \frac{f_n(\xi)}{(\xi - z)^2} d\xi - \int_{|\xi - z| = \rho} \frac{f(\xi)}{(\xi - z)^2} d\xi \right| \\
&= \left| \int_{|\xi - z| = \rho} \frac{f_n(\xi) - f(\xi)}{(\xi - z)^2} d\xi \right| \\
&\leq \int_{|\xi - z| = \rho} \frac{|f_n(\xi) - f(\xi)|}{|\xi - z|} d|\xi| \\
&\leq \int_0^{2\pi} \frac{\varepsilon}{\rho} d\theta \\
&= \frac{2\pi\varepsilon}{\rho}
\end{aligned}$$

for all $n \geq N$ and so $f_n' \to f$ for all $z \in K$.

(c) Let $f(z) = 0$ for some z. Let $\rho > 0$ be such that $B_\rho(z)$ contains only one zero of f, namely z itself. Then from (b), $f_n' \to f'$, so

$$\begin{aligned}
0 &= \text{ Number of zeros of } f_n \text{ in } B_\rho(z) \\
&= \frac{1}{2\pi i}\int_{|\xi - z| = \rho} \frac{f_n(\xi)}{f_n'(\xi)} d\xi \to \frac{1}{2\pi i}\int_{|\xi - z| = \rho} \frac{f(\xi)}{f'(\xi)} d\xi \\
&= \text{ Number of zeros of } f \text{ in } B_\rho(z) \\
&= 1
\end{aligned}$$

a contradiction unless $f \equiv 0$ in which case,

$$\frac{1}{2\pi i}\int_{|\xi - z| = \rho} \frac{f_n(\xi)}{f_n'(\xi)} d\xi \to 0.$$

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Problem 4.

(a) Suppose that f is analytic on the open unit disk $\{z : |z| < 1\}$ and that there exists a constant M such that $|f^k(0)| \leq k^4 M^k$ for all $k \geq 0$. Show that f can be extended to be analytic on $\mathbb{C}$.

(b) Suppose that f is analytic on the open unit disk $\{z : |z| < 1\}$ and that there exists a cosntant $M > 1$ such that $|f(1/k)| \leq M^{-k}$ for all $k \geq 1$. Show that f is identically zero.

Solution.

(a) Note that this forces $f(0) = 0$ and so the statement as written, makes no sense since $f^k(0) = 0$ for all k.

Thus, $|f^{(k)}(0)| \leq k^4 M^k$ for all $k \geq 0$.

We can write a Taylor Series for f inside the disk.

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n.$$

Now, for arbitrary $z \in \mathbb{C}$,

$$\begin{aligned}|f(z)| &\leq \sum_{n=0}^{\infty} \frac{|f^{(n)}(0)|}{n!}|z|^n \\ &\leq \sum_{n=0}^{\infty} \frac{n^4 M^n}{n!}|z|^n \\ &= \sum_{n=0}^{\infty} \frac{n^4 R^n}{n!} \qquad R = M|z|\end{aligned}$$

This is a real sum, and we can check that it converges by ratio test. Namely, $\sum_{n=0}^{\infty} a_n$ converges absolutely if

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < 1$$

$$\begin{aligned}\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{\frac{(n+1)^4 R^{n+1}}{(n+1)!}}{\frac{n^4 R^n}{n!}} \\ &= \lim_{n\to\infty} \frac{(n+1)^4 R^{n+1} n!}{(n+1)! n^4 R^n} \\ &= \lim_{n\to\infty} \frac{(n+1)^4 R}{n^4 (n+1)} \\ &= \lim_{n\to\infty} \frac{(n+1)^3 R}{n^4} = 0\end{aligned}$$

Thus, the series absolutely converges. Furthermore, for all $|z| < \frac{R}{M}$, convergence is clearly uniform and since R is arbitrary, we get that f can be extended analytically to all of $\mathbb{C}$.

(b) Suppose that f is analytic on the open unit disk $\{z : |z| < 1\}$ and that there exists a constant $M > 1$ such that $|f(1/k)| \leq M^{-k}$ for all $k \geq 1$.

This gives that $f(0) = 0$. Assume f is non-constant. Then we can write $f(z) = z^n g(z)$ where g is analytic and nonzero at 0.

Then

$$|f(1/k)| = \frac{|g(1/k)|}{k^n} \leq \frac{1}{M^k}.$$

Namely,

$$\lim_{k\to\infty} |g(1/k)| \leq \lim_{k\to\infty} \frac{k^n}{M^k} = 0$$

by induction.

This contradicts that g is nonzero at 0 and so g cannot exist. Namely, f cannot exist and must therefore be constant.

Since $f(0) = 0$, $f \equiv 0$.

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