# Kayla Orlinsky

## Complex Analysis Exam Fall 2014

**Problem 1.** Let $a > 1$. Compute

$$\int_0^\pi \frac{d\theta}{a + \cos\theta}$$

being careful to justify your methods.

**Solution.** Note that $\cos(\theta) = \cos(\theta - 2\pi)$ and $\cos(-\theta) = \cos(\theta)$ so

$$\begin{aligned}
\int_\pi^{2\pi} \frac{d\theta}{a+\cos(\theta)} &= \int_\pi^{3\pi/2} \frac{d\theta}{a+\cos(\theta)} + \int_{3\pi/2}^{2\pi} \frac{d\theta}{a+\cos(\theta)} \\
&= \int_{-\pi}^{-\pi/2} \frac{d\theta}{a+\cos(\theta)} + \int_{-\pi/2}^{0} \frac{d\theta}{a+\cos(\theta)} \\
&= \int_{\pi}^{\pi/2} \frac{-d\theta}{a+\cos(\theta)} + \int_{\pi/2}^{0} \frac{-d\theta}{a+\cos(\theta)} \\
&= \int_{\pi/2}^{\pi} \frac{d\theta}{a+\cos(\theta)} + \int_{0}^{\pi/2} \frac{d\theta}{a+\cos(\theta)} \\
&= \int_{0}^{\pi} \frac{d\theta}{a+\cos(\theta)}
\end{aligned}$$

and so

$$\int_0^{2\pi} \frac{d\theta}{a+\cos\theta} = 2\int_0^\pi \frac{d\theta}{a+\cos(\theta)}.$$

Thus,

$$\begin{aligned}
\int_0^\pi \frac{d\theta}{a+\cos\theta} &= \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{a+\cos(\theta)} \\
&= \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{a+\frac{e^{i\theta}+e^{-i\theta}}{2}} \\
&= \frac{1}{2}\int_0^{2\pi} \frac{2d\theta}{2a+e^{i\theta}+e^{-i\theta}} \\
&= \int_0^{2\pi} \frac{e^{i\theta}d\theta}{e^{2i\theta}+2ae^{i\theta}+1} \\
&= \int_{|z|=1} \frac{-idz}{z^2+2az+1} \qquad z=e^{i\theta} \\
&= \int_{|z|=1} \frac{-idz}{(z+a+\sqrt{a^2-1})(z+a-\sqrt{a^2-1})} \qquad (1) \\
&= 2\pi i \mathrm{Res}_{z=-a+\sqrt{a^2-1}} \frac{-i}{(z+a+\sqrt{a^2-1})(z+a-\sqrt{a^2-1})} \qquad \text{Residue Theorem} \qquad (2) \\
&= 2\pi \frac{1}{-a+\sqrt{a^2-1}+a+\sqrt{a^2-1}} \\
&= \frac{2\pi}{2\sqrt{a^2-1}} \\
&= \frac{\pi}{\sqrt{a^2-1}}
\end{aligned}$$

with (1) since

$$\begin{aligned}
z^2+2az+1 &= 0 \\
\implies z &= \frac{-2a\pm\sqrt{4a^2-4}}{2} \\
&= -a\pm\sqrt{a^2-1}
\end{aligned}$$

and (2) since $a+\sqrt{a^2-1}>a>1$ this point is not in the circle $\{|z|<1\}$. Furthermore,

$$0=a-a<a-\sqrt{a^2-1} \qquad \text{since } a>1 \implies \sqrt{a^2-1}<\sqrt{a^2}=a$$

and so if $h(a)=a-\sqrt{a^2-1}$ then $h'(a)=1-\frac{a}{\sqrt{a^2-1}}<0$ so $h$ is strictly decreasing and since $h(1)=1$, we have that $0<a-\sqrt{a^2-1}<1$ for all $a>1$, so this is in the disk $\{|z|<1\}$.

✌

**Problem 2.** Find the number of zeros, counting multiplicity, of $z^8 - z^3 + 10$ inside the first quadrant $\{z \in \mathbb{C} : \mathfrak{Re}(z) > 0, \mathfrak{Im}(z) > 0\}$.

**Solution.** Let $f(z) = z^8 - z^3 + 10$.

First, on $\mathfrak{Re}(z) = x > 0$, $f(x) = x^8 - x^3 + 10 > 0$.

This is because if $0 < x < 1$ then $x^8 < x^3$ so $x^8 - x^3 > -x^3 > -1$ and so $f(x) > 9$.

And if $x > 1$ then $x^8 > x^3$ so $f(x) > 0$. So $f$ maps the positive real axis to the positive real axis away from the origin.

Similarly,

$$f(iy) = (iy)^8 - (iy)^3 + 10 = y^8 + 10 + iy^3$$

and so $\mathfrak{Re}(f(iy)) > 0$ and $\mathfrak{Im}(f(iy)) > 0$ for all $y > 0$ so $f$ sends the positive imaginary axis to the first quadrant away from the origin.

Finally, for $R$ large, we can define a quarter circle arc in the first quadrant $\{z = Re^{i\theta} : 0 \leq \theta \leq \frac{\pi}{2}\}$. On this arc,

$$\lim_{R\to\infty} \frac{f(Re^{i\theta})}{R^8} = \lim_{R\to\infty} \left(e^{8i\theta} - \frac{e^{3i\theta}}{R^5} + \frac{10}{R^8}\right) = e^{8i\theta}.$$

Thus, $f$ has a total change in argument of $8\pi/2 = 4\pi$. Namely, $f$ has 2 roots in the first quadrant. ✌

**Problem 3.** Assume that $f(z)$ and $g(z)$ are holomorphic in a puctured neighborhood of $z_0 \in \mathbb{C}$. Prove that if $f$ has an essential singuliarty at $z_0$ and $g$ has a pole at $z_0$, then $f(z)g(z)$ has an essential singulairty at $z_0$.

**Solution.** Let $h(z) = f(z)g(z)$. Then since $g(z)$ has a pole at $z_0$,

$$\frac{1}{g}$$

has a zero at $z_0$. Thus, if $h$ is analytic at $z_0$ then $\frac{h}{g}$ has a zero at $z_0$. However, since $\frac{h}{g} = f$ which has an essential singularity at $z_0$, this is not possible.

So $h$ has a singularity at $z_0$.

Now, if $h$ has a removable singularity, then $\frac{1}{g}$ having a zero at $z_0$ implies

$$\lim_{z\to z_0} \frac{h}{g} = 0 \neq \lim_{z\to z_0} f(z)$$

so this is a contradiction.

If $h$ has a pole at $z_0$, then $h = \frac{h'}{(z-z_0)^k}$ and since $\frac{1}{g}$ has a zero at $z_0$, $\frac{1}{g} = (z-z_0)^l g'$ for some $k$, some $l$, some $h'$ which is non-zero at $z_0$ and some $g'$ which is nonzero at $z_0$.

However, then if $l < k$

$$\begin{aligned}\lim_{z\to z_0}(z-z_0)^{k-l+1}\frac{h}{g} &= \lim_{z\to z_0}(z-z_0)^{k-l+1}\frac{h'(z-z_0)^l g'}{(z-z_0)^k}\\ &= \lim_{z\to z_0}(z-z_0)h'g'\\ &= 0\\ &\neq \lim_{z\to z_0}(z-z_0)^{k-l+1}f(z)\end{aligned}$$

since $f$ has an essential singuliarty.

And if $l \geq k$, then

$$\lim_{z\to z_0}\frac{h}{g} = \lim_{z\to z_0}(z-z_0)^{l-k}h'g' < \infty$$

and again we get a contradiction since $\frac{h}{g} = f$ which has an essential singuliarty at $z_0$ and so cannot possess a finite limit at that point.

Therefore, $h$ has a singularity which not removable and not a pole. Namely, $h$ must have an essential singularity. ✌

**Problem 4.**

(a) Suppose that $f$ is holormophic on $\mathbb{C}$ and assume that the imaginary part of $f$ is bounded. Prove that $f$ is constant.

(b) Suppose that $f$ and $g$ are holomorphic on $\mathbb{C}$ and that $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{C}$. Prove that there exists $\lambda \in \mathbb{C}$ such that $f = \lambda g$.

**Solution.**

(a) Suppose that $f$ is holormophic on $\mathbb{C}$ and assume that the imaginary part of $f$ is bounded.

Let $f = u + iv$. Assume $|v(z)| < M$ for all $z \in \mathbb{C}$.

Thus, $f(\mathbb{C}) \subset \Omega = \{x + iy : |y| < M\}$. Since $\Omega$ is an open simply connected strict subset of $\mathbb{C}$, by the Riemann Mapping Theorem, there exists a $g : \Omega \to \mathbb{D} = \{|z| < 1\}$ which is analytic, bijective, and has an analytic inverse.

However, then

$$g \circ f : \mathbb{C} \to \mathbb{D}$$

is an entire function which is bounded and so $g \circ f = c$ is constant. Namely, $f = g^{-1}(c)$ and is constant.

(b) Suppose that $f$ and $g$ are holomorphic on $\mathbb{C}$ and that $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{C}$.

Assume $g$ has a zero of order $k$ at $z_0$. Then $g(z) = (z - z_0)^k g_0(z)$ with $g_0$ analytic and nonzero in a neighborhood of $z_0$.

However, then

$$0 = \lim_{z \to z_0} \frac{|g(z)|}{(z - z_0)^{k-1}} \geq \lim_{z \to z_0} \frac{|f(z)|}{(z - z_0)^{k-1}}$$

and so $f$ must have a zero of order at least $k$ at $z_0$.

Namely, if $h = \frac{f}{g}$ then $h$ has removable singularities at the zeros of $g$, and so has an analytic continuation to the whole plane.

Namely, WLOG, $h$ is entire, and since

$$|h| = \frac{|f|}{|g|} \leq 1$$

by Louiville's $h$ is consant. So there is some $\lambda \in \mathbb{C}$ so $\frac{f}{g} = \lambda$ and namely, $f = \lambda g$.

✌