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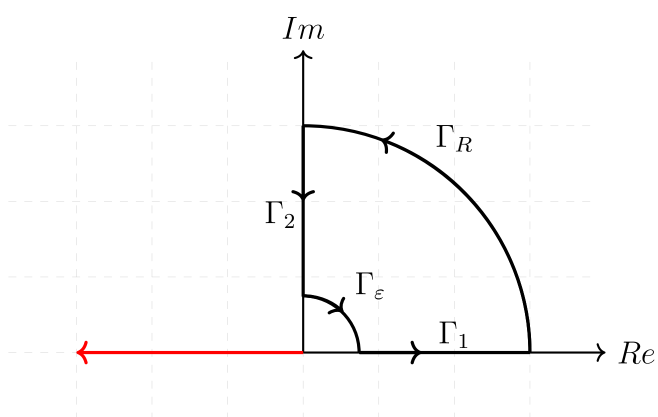
Complex Analysis Exam Spring 2013

Problem 1. Evaluate
$$\int_0^\infty \frac{x^{1/3}}{1+x^4}dx$$
being careful to justify your answer.

Solution. We will use the contour around the top right quadrant avoiding the origin with the principal branch for $\frac{x^{1/3}}{1+x^4}$ being $(-\infty, 0]$.



Let

$$I_1 = \int_{\Gamma_1} \frac{z^{1/3}}{1+z^4}dz$$
$$I_2 = \int_{\Gamma_2} \frac{z^{1/3}}{1+z^4}dz$$
$$I_\varepsilon = \int_{\Gamma_\varepsilon} \frac{z^{1/3}}{1+z^4}dz$$
$$I_R = \int_{\Gamma_R} \frac{z^{1/3}}{1+z^4}dz$$

Now,

$$\begin{aligned} I_2 &= \int_R^\varepsilon \frac{(ix)^{1/3} i}{1+x^4} dx \\ &= \int_R^\varepsilon \frac{ie^{\frac{\pi i}{6}} x^{1/3}}{1+x^4} dx \\ &= -ie^{\frac{\pi i}{6}} \int_\varepsilon^R \frac{x^{1/3}}{1+x^4} dx \\ &= -ie^{\frac{\pi i}{6}} I_1 \end{aligned}$$

Now,

$$\begin{aligned} |I_\varepsilon| &= \left| \int_{\Gamma_\varepsilon} \frac{z^{1/3}}{1+z^4} dz \right| \\ &\leq \int_{|z|=\varepsilon} \frac{\varepsilon^{1/3}}{|1+z^4|} |dz| \\ &\leq \int_{|z|=\varepsilon} \frac{\varepsilon^{1/3}}{1-\varepsilon^4} |dz| \\ &= -\pi\varepsilon \frac{\varepsilon^{1/3}}{1-\varepsilon^4} \to 0 \qquad \varepsilon \to 0. \end{aligned}$$

and similarly,

$$\begin{aligned} |I_R| &\leq -\pi R \frac{R^{1/3}}{1-R^4} \\ &= -\pi \frac{R^{4/3}}{1-R^4} \to 0 \qquad R \to \infty \end{aligned}$$

Finally, we note that there is one residue in the first quadrant since

$$1+x^4 = (i-x^2)(i+x^2) = (\sqrt{i}-x)(\sqrt{i}+x)(i\sqrt{i}+x)(i\sqrt{i}-x) \qquad \sqrt{i} = e^{i\pi/4} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$$

Alternatively, the zeros are $e^{ki\pi/4}$ for $k = 1, 3, 5, 7$ for which only the first is in the first quadrant.

Therefore, by the Residue Theorem, we get

$$\begin{aligned}
2\pi i \text{Res}_{z=e^{i\pi/4}} \frac{z^{1/3}}{1+z^4} &= 2\pi i \lim_{z\to e^{i\pi/4}} \frac{z^{1/3}(z-e^{i\pi/4})}{1+z^4} \\
&= 2\pi i \lim_{z\to e^{i\pi/4}} \frac{z^{4/3}-e^{i\pi/4}z^{1/3}}{1+z^4} \\
&= 2\pi i \lim_{z\to e^{i\pi/4}} \frac{\frac{4}{3}z^{1/3}-\frac{1}{3}e^{i\pi/4}z^{-2/3}}{4z^3} && \text{L'Hopital's Rule} \\
&= 2\pi i \frac{\frac{4}{3}e^{i\pi/12}-\frac{1}{3}e^{i\pi/4}e^{-2i\pi/12}}{4e^{3i\pi/4}} \\
&= 2\pi i \frac{\frac{4}{3}e^{i\pi/12}-\frac{1}{3}e^{i\pi/12}}{4e^{3i\pi/4}} \\
&= 2\pi i \frac{e^{i\pi/12}}{4e^{3i\pi/4}} \\
&= i\frac{\pi}{2}e^{-8i\pi/12} \\
&= i\frac{\pi}{2}e^{-2i\pi/3} \\
&= i\frac{\pi}{2}e^{4i\pi/3} \\
&= \lim_{R\to\infty}\lim_{\varepsilon\to 0}(I_1+I_2+I_\varepsilon+I_R) \\
&= (1-ie^{\frac{\pi i}{6}})\int_0^\infty \frac{x^{1/3}}{1+x^4}dx \\
\implies \int_0^\infty \frac{x^{1/3}}{1+x^4}dx &= \frac{i\frac{\pi}{2}e^{4i\pi/3}}{1-e^{4\pi i/6}} \\
&= \frac{i\pi e^{4i\pi/3}}{2(1-e^{2\pi i/3})} \\
&= \frac{i\pi}{2(e^{-4i\pi/3}-e^{-2i\pi/3})} \\
&= \frac{\pi}{4}\frac{i2}{e^{2i\pi/3}-e^{-2i\pi/3}} \\
&= \frac{\pi/4}{\sin(2\pi/3)} \\
&= \frac{\pi/4}{\sqrt{3}/2} \\
&= \frac{\pi}{2\sqrt{3}}
\end{aligned}$$

✌

Problem 2. Assume that f is an entire function such that

$$|f(z)| \geq \frac{1}{1+|z|} \qquad \text{for all } z \in \mathbb{C}.$$

Prove that f is a constant function.

Solution. First, we note that $f(z) \neq 0$ for all $|z| < \infty$ since $\frac{1}{1+|z|} > 0$.

Thus, because f is entire and non-zero, $g = \frac{1}{f}$ is entire and non-zero.

Thus,

$$|g(z)| \leq 1 + |z|.$$

Therefore, by Cauchy Estimate, on $\{|z| \leq R\}$,

$$|g(z)| \leq 1 + |z| \leq 1 + R$$

and so

$$|g^{(2)}(z)| \leq \frac{2!(1+R)}{R^2} \to 0 \qquad R \to \infty.$$

Thus, $g(z) = az + b$, however, then $g(z)$ has a zero at $\frac{-b}{a}$ which is a contradiction unless $a = 0$.

Namely, g is constant and so f is constant. ✌

Problem 3. Let f_n, $n \geq 1$, be a sequence of holomorphic functions on an open connected set D such that $|f_n(z)| \leq 1$ for all $z \in D$, $n \geq 1$. Let $A \subset D$ be the set of all $z \in D$ for which the limit $\lim_n f_n(z)$ exists.

Show that if A has an accumulation point in D, then there exists a holomoprhic function f on D such that $f_n \to f$ uniformly on every compact subset of D as $n \to \infty$.

Solution. Assume A has an accumulation point in $z_0 \in D$.

By Montel's theorem, since the f_n are all uniformly bounded on D, $\{f_n\}$ form a normal family on D.

Thus, on every compact subset of D, there exists a holomorphic function f such that there is a subsequence $\{f_{n_k}\}$ which converges uniformly to f.

We would like to show that the entire sequence $f_n \to f$ on each compact subset.

Let $f_{n_k} \to f$ on a compact subset $K \subset D$ containing z_0 which is the accumulation point of A in D. Let $g_n = f_n - f$.

Then $g_{n-k} \to 0$ uniformly on K. Therefore, $g_n \to 0$ on $K \cap A$ since the limit exists and a subsequence converges to 0.

Now, since $A \cap K$ contains an accumulation point, there is a sequence $\{z_l\} \subset A \cap K$ such that $z_l \to z_0$ and $\lim_{n\to\infty} g_n(z_l) = 0$ for all l.

Now, since $\{f_n\}$ is uniformly bounded and holomorphic, by Arzela-Ascoli it is equicontinuous and so $\{g_n\}$ is equicontinuous. Let $\varepsilon > 0$ be given and $\delta > 0$ be such that $|g_n(z) - g_n(w)| < \varepsilon$ whenever $|z - w| < \delta$.

Therefore, if $\{B_{\delta/3}(z_l)\}$ is an open covering of some compact subset of K, then there exists a finite subcovering $B_{\delta/3}(z_{l_j})$. Now, since $g_n \to 0$ pointwise on K, for each z_{l_j} take N_j such that $|g_n(z_{l_j})| < \varepsilon$ for all $n \geq N_j$.

Let N be the maximimum of the N_j. Then for z in this finite subcovering, there is a z_{l_j} such that $|z - z_{l_j}| < \delta$ and so

$$|g_n(z)| \leq |g_n(z) - g_n(z_{l_j})| + |g_n(z_{l_j})| < \varepsilon + \varepsilon$$

for all $n \geq N$.

Thus, $g_n \to 0$ uniformly on some compact subset of K. However, this can clearly be extended to all of K and therefore to any compact subset of D using similar open coverings.

and so $g_n(z) \to 0$ uniformly on compact subsets of D. Thus, $f_n \to f$ uniofmrly on compact subsets of D. ✌

Problem 4. Let $f(z)$ be meromorphic on $\mathbb{C}$, holomoprhic for $Rez > 0$ and such that $f(z+1) = zf(z)$ in its domain with $f(1) = 1$. Show that f has the first order poles at $0, -1, -2, ...$, and find the residues of f at these points.

Solution. $f(1) = f(0+1) = 1$ implies that

$$\lim_{z\to 0} zf(z) = 1.$$

Namely, that f has a first order pole (else the limit would be 0 or infinity) at 0 of residue 1.

Now,

$$\lim_{z\to -1}(z+1)f(z) = \lim_{z\to -1}(z+1)\frac{f(z+1)}{z} = \lim_{w\to 0}\frac{wf(w)}{w-1} = \frac{1}{-1} = -1$$

and so f has a first order pole at -1 of residue -1.

Similarly, since

$$f(z) = \frac{f(z+1)}{z} = \frac{f(z+2)}{z(z+1)} = \frac{f(z+n)}{z(z+1)\cdots(z+n-1)}$$

we get that

$$\lim_{z\to -n}(z+n)f(z) = \lim_{z\to -n}\frac{(z+n)f(z+n)}{z(z+1)\cdots(z+n-1)} = \lim_{w\to 0}\frac{wf(w)}{(w-n)(w-n+1)\cdots(w-1)} = \frac{1}{(-n)(-n+1)\cdots}$$

Since all these poles are first order (as evidenced by the limit), these limits correspond exactly to the residues and so f has first order poles at $-n$ for all $n \in \mathbb{N}$ with residue $\frac{1}{(-n)(-n+1)\cdots(-1)}$ and a first order pole at 0 with residue 1. ✌