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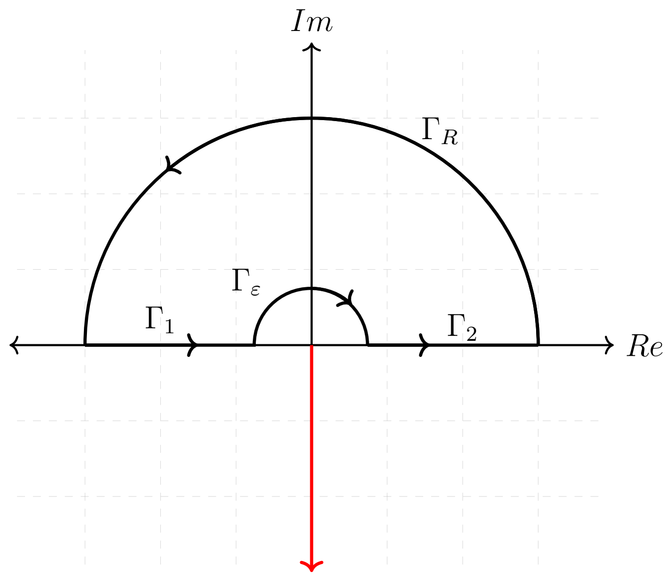
Complex Analysis Exam Fall 2013

Problem 1. Compute
$$\int_0^\infty \frac{\log^2 x}{1+x^2}dx$$

Solution. We will use "Ol' Faithful" the contour around the upper half plane avoiding the origin since every branch cut of $\log x$ intersects 0.

Then we take any branch which does not intersect the upper half plane (including the real line).



Let

$$I_1 = \int_{\Gamma_1} \frac{\log^2 z}{z^2+1}dz$$
$$I_2 = \int_{\Gamma_2} \frac{\log^2 z}{z^2+1}dz$$
$$I_\varepsilon = \int_{\Gamma_\varepsilon} \frac{\log^2 z}{z^2+1}dz$$
$$I_R = \int_{\Gamma_R} \frac{\log^2 z}{z^2+1}dz$$

Note that

$$\begin{aligned} I_1 &= \int_{-R}^{-\varepsilon} \frac{\log^2 x}{1+x^2}dx \\ &= \int_R^{\varepsilon} \frac{-(\log x + \pi i)^2}{1+x^2}dx \\ &= \int_{\varepsilon}^R \frac{\log^2 x + 2\pi i \log x - \pi^2}{1+x^2}dx \\ &= I_2 + 2\pi i \int_{\varepsilon}^R \frac{\log x}{1+x^2}dx - \pi^2 \int_{\varepsilon}^R \frac{1}{1+x^2}dx \\ &= I_2 - \pi^2(\tan^{-1}(R) - \tan^{-1}(\varepsilon)) + 2\pi i \int_{\varepsilon}^R \frac{\log x}{1+x^2}dx \end{aligned}$$

Now,

$$\begin{aligned} |I_R| &= \left| \int_{\Gamma_R} \frac{\log^2 z}{1+z^2}dz \right| \\ &\leq \int_0^{\pi} \frac{R|\log R + i\theta|^2}{R^2-1}d\theta \\ &\leq \pi \frac{R\log^2 R + 2R\pi \log R + R\pi^2}{R^2-1} \to 0 \qquad R \to \infty \end{aligned}$$

since

$$\lim_{R\to\infty} \frac{\log^2 R}{R} = \lim_{R\to\infty} \frac{2\log R}{R} = \lim_{R\to\infty} \frac{2}{R} = 0$$

by L'Hopital's Rule and similarly, $\frac{\log R}{R} \to 0$.

Similarly,

$$\begin{aligned} |I_{\varepsilon}| &\leq \int_{\pi}^0 \frac{\varepsilon|\log \varepsilon + i\theta|^2}{\varepsilon^2-1}d\theta \\ &\leq \pi \frac{\varepsilon \log^2 \varepsilon + 2\varepsilon\pi \log \varepsilon + \varepsilon\pi^2}{\varepsilon^2-1} \to 0 \qquad \varepsilon \to 0 \end{aligned}$$

since

$$\lim_{\varepsilon\to 0} \varepsilon \log^2 \varepsilon = \lim_{\varepsilon\to 0} \frac{2\log \varepsilon}{\frac{-1}{\varepsilon}} = \lim_{\varepsilon\to 0} \frac{2}{\frac{1}{\varepsilon}} = 0$$

by L'Hopital's Rule.

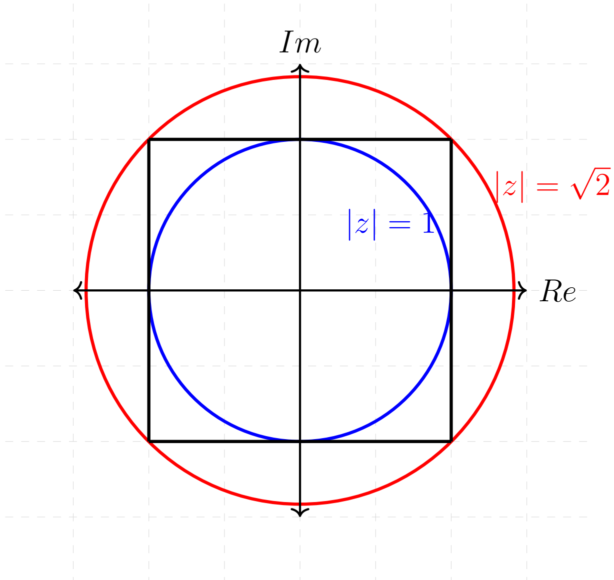
Thus, by the Residue Theorem,

$$\begin{aligned}
2\pi i \text{Res}_{z=i} \frac{\log^2 z}{z^2+1} &= 2\pi i \frac{\log^2(i)}{i+i} \\
&= \pi \left(i\frac{\pi}{2} \right)^2 \\
&= -\frac{\pi^3}{4} \\
&= \lim_{R\to\infty} \lim_{\varepsilon\to 0} (I_1 + I_2 + I_\varepsilon + I_R \\
&= 2\int_0^\infty \frac{\log^2 x}{1+x^2} dx - \frac{\pi^3}{2} \\
\implies \int_0^\infty \frac{\log^2 x}{1+x^2} dx &= \frac{\pi^3}{8}
\end{aligned}$$

Note that since the residue is real this forces $\int_0^\infty \frac{\log x}{1+x^2} dx = 0$. ✌

Problem 2. Find the number of *distinct* zeros of $f(z) = z^6 + (10-i)z^4 + 1$ inside $(-1,1)\times(-1,1)$.

Solution. Let $g(z) = -z^6$. We will inscribe a circle in the unit square and inscribe the square in a circle.



Then on $\{|z| = \sqrt{2}\}$

$$\begin{aligned}|f(z)| &\geq |10-i||z|^4 - |z|^6 - 1\\ &= \sqrt{101}(\sqrt{2})^4 - (\sqrt{2})^6 - 1\\ &= 4\sqrt{101} - 8 - 1\\ &\geq 40 - 9\\ &= 31\\ &> 8\\ &= |-z|^6\\ &= |g(z)|\end{aligned}$$

and so $|f(z)| > |g(z)|$ so $f(z)$ and $f(z)+g(z)$ have the same number of zeros inside $\{|z| \leq \sqrt{2}\}$.

Since $f(z) + g(z) = (10-i)z^4 + 1$, we need only count the number of zeros of this function. Since if

$$(10-i)z^4 + 1 = 0 \implies z^4 = \frac{1}{i-10} \implies |z|^4 = \frac{1}{|i-10|} = \frac{1}{\sqrt{101}} < 4$$

and so $|z| < \sqrt{2}$. Namely, the zeros of $f+g$ all lie in $\{|z| \leq \sqrt{2}\}$ and so $f+g$ and f have four zeros in $\{|z| \leq \sqrt{2}\}$.

Now, on $\{|z|=1\}$

$$\begin{aligned}|f(z)| &\geq |10-i||z|^4 - |z|^6 - 1\\ &= \sqrt{101} - 1 - 1\\ &= \sqrt{101} - 2\\ &> 10 - 2\\ &= 8\\ &> 8\\ &= |-z|^6\\ &= |g(z)|\end{aligned}$$

And so again, $f(z)$ and $f(z)+g(z)$ have the same number of zeros on $|z|=1$.

Namely, since we already saw f has 4 zeros in $\{|z|\leq\sqrt{2}\}$ and 4 zeros in $\{|z|\leq 1\}$, we have that f has 4 zeros in the unit square.

Now, we need only show that the zeros of f are all unique.

If f has any repeated roots in the unit square, then f and f' would have a root in common.

Since $f'(z) = 6z^5 + 4(10-i)z^3$ we get that $f'(z) = 0$ implies that $2z^3(3z^2 + 2(10-i)) = 0$ and since 0 is not a root of f,

the only possibilities are when $z^2 = \frac{2i-20}{3}$. However, clearly neither of these roots lie in the unit square, since the largest magnitude inside the unit square is $\sqrt{2}$ (so $|z|^2 \leq 2$) and

$$|\frac{2i-20}{3}| = \frac{\sqrt{2}}{\sqrt{3}}\sqrt{101} > \frac{1}{2}10 = 5 > 2.$$

Thus, f and f' share no zeros in the unit square and so f has 4 distinct zeros inside the unit square. ✌

Problem 3. Suppose that f is holomorphic in a neighborhood U of $a \in \mathbb{C}$. Consider the following two statements:

(i) There exist two sequences $\{z_k\}_{k=1}^{\infty}$ and $\{w_k\}_{k=1}^{\infty}$ in $U \backslash \{a\}$ converging to a such that $z_k \neq w_k$ and $f(z_k) = f(w_k)$ for all $k \in \mathbb{N}$.

(ii) $f'(a) = 0$.

Determine whether either of the statements implies the other one. In each case jusifty your answer with a proof or counterexample.

Solution. $\boxed{(i) \implies (ii)}$ True. Assume $f'(z) \neq 0$. Then because f is analytic, the inverse function theorem states that f is invertible in a small neighborhood of a. Namely, f must be injective on a small neighborhood of a and so the sequences $\{z_k\}$ and $\{w_k\}$ cannot exist.

Thus, $f'(a) = 0$.

$\boxed{(i) \impliedby (ii)}$ Since $f'(a) = 0$ WLOG we may take $f(a) = 0$ (else we look at $g(z) = f(z) - f(a)$).

Then we can write $f(z) = (z - a)^n h(z)$ where $n \geq 2$ $h(z)$ is analytic in U and nonzero in a neighborhood of a.

Since h is analytic, after picking a branch, we can write $f(z) = (g(z))^n$ where $g(z) = (z - a)h^{1/n}(z)$ and $h^{1/n}$ is also analytic and nonzero in a neighborhood of a.

Now, by the open mapping theorem, $f(U)$ is open in $\mathbb{C}$ and $0 \in f(U)$. Thus, there exists a neighborhood V of 0 such that $V \subset f(U)$.

Namely, $(g(U))^{1/n}$ contains a neighborhood of 0 and so $re^{2k\pi i/n} \in (g(U))^{1/n}$ for some $r > 0$ and $1 \leq k \leq n$.

Namely, g cannot be injective since it wraps some neighborhood of a around the origin n-times. Thus, f is also not injective in a neighborhood of a and so (i) is true. ✌

Problem 4. Let f be analytic in an open set $U \subset \mathbb{C}$, and let $K \subset U$ be compact. Show that there exists a constant C depending on U and K such that

$$|f(z)| \leq C \left(\int_U |f|^2 \right)^{1/2}$$

Solution. Let $\{B_r(z)\}_{z \in K}$ be an open cover of K. Then by the Lebesgue number lemma, there exists $\delta > 0$ such that $B_\delta(z) \subset B_r(z')$ for some $z' \in K$.

$$\begin{aligned}
|f(z)| &= \left| \frac{1}{2\pi i} \int_{|\xi - z| = \delta} \frac{f(\xi)}{\xi - z} d\xi \right| \\
&\leq \frac{1}{2\pi} \int_{|\xi - z| = \delta} \frac{|f(\xi)|}{|\xi - z|} d|\xi| \\
&\leq \frac{1}{2\pi} \left(\int_{|\xi - z| = \delta} |f(\xi)|^2 d|\xi| \right)^{1/2} \left(\int_{|\xi - z| = \delta} \frac{1}{|\xi - z|^2} d|\xi| \right)^{1/2} && \text{Holder's Inequality} \\
&= \frac{1}{2\pi} \left(\int_{|\xi - z| = \delta} |f(\xi)|^2 d|\xi| \right)^{1/2} \left(\int_0^{2\pi} \frac{1}{\delta} d\theta \right)^{1/2} \\
&= \frac{1}{2\pi} \left(\int_{|\xi - z| = \delta} |f(\xi)|^2 d|\xi| \right)^{1/2} \frac{\sqrt{2\pi}}{\sqrt{\delta}} \\
&= \frac{1}{\sqrt{2\pi\delta}} \left(\int_{|\xi - z| = \delta} |f(\xi)|^2 d|\xi| \right)^{1/2}
\end{aligned}$$

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