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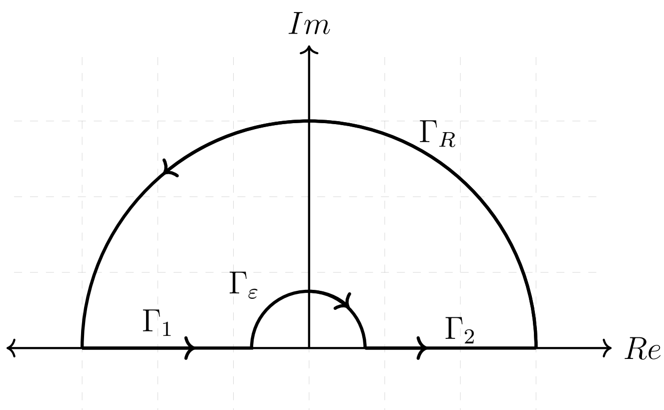
Complex Analysis Exam Spring 2012

Problem 1. Suppose $a > 0$. Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\sin(ax)}{x(x^2+1)} dx$$

be careful to justify your methods.

Solution. We will use "Ol' Faithful" the contour around the upper half plane avoiding the origin.



Let

$$I_1 = \int_{\Gamma_1} \frac{e^{iaz}}{z(z^2+1)} dz$$

$$I_2 = \int_{\Gamma_2} \frac{e^{iaz}}{z(z^2+1)} dz$$

$$I_\varepsilon = \int_{\Gamma_\varepsilon} \frac{e^{iaz}}{z(z^2+1)} dz$$

$$I_R = \int_{\Gamma_R} \frac{e^{iaz}}{z(z^2+1)} dz$$

Now,

$$\begin{aligned}
I_1 + I_2 &= \int_{-R}^{-\varepsilon} \frac{e^{iax}}{x(x^2+1)} dx + \int_{\varepsilon}^{R} \frac{e^{iax}}{x(x^2+1)} dx \\
&= \int_{R}^{\varepsilon} \frac{-e^{-iax}}{-x((-x)^2+1)} dx + \int_{\varepsilon}^{R} \frac{e^{iax}}{x(x^2+1)} dx \\
&= \int_{\varepsilon}^{R} \frac{-e^{-iax}}{x(x^2+1)} dx + \int_{\varepsilon}^{R} \frac{e^{iax}}{x(x^2+1)} dx \\
&= \int_{\varepsilon}^{R} \frac{e^{iax} - e^{-iax}}{x(x^2+1)} dx \\
&= \int_{\varepsilon}^{R} \frac{2i\sin(ax)}{x(x^2+1)} dx
\end{aligned}$$

Since $\sin x$ is an odd function (on the real line), we have that

$$\int_{-\infty}^{0} \frac{\sin(ax)}{x(x^2+1)} dx = \int_{\infty}^{0} \frac{-\sin(-ax)}{-x(x^2+1)} dx = \int_{0}^{\infty} \frac{-\sin(-ax)}{x(x^2+1)} dx = \int_{0}^{\infty} \frac{\sin(ax)}{x(x^2+1)}.$$

Now, we note that $\frac{e^{iaz}}{z(z^2+1)}$ has an isolated pole of order 1 at $z = 0$ since

$$\lim_{z\to 0} z \frac{e^{iaz}}{z(z^2+1)} = \lim_{z\to 0} \frac{e^{iaz}}{z^2+1} = \frac{1}{1} = 1.$$

Therefore, we can write $\frac{e^{iaz}}{z(z^2+1)} = \frac{1}{z} + f(z)$ where $f(z)$ is analytic at 0. Thus, taking ε small, we get

$$\begin{aligned}
I_\varepsilon &= \int_{\Gamma_\varepsilon} \frac{e^{iaz}}{z(z^2+1)} dz \\
&= \int_{\Gamma_\varepsilon} \frac{1}{z} + f(z) dz \\
&= \int_{\pi}^{0} i d\theta + 0 \\
&= -i\pi.
\end{aligned}$$

Now,

$$
\begin{aligned}
|I_R| &= \left|\int_{\Gamma_R} \frac{e^{iaz}}{z(z^2+1)}dz\right| \\
&= \left|\int_0^\pi \frac{ie^{iaRe^{i\theta}}}{R^2e^{2i\theta}+1}d\theta\right| \\
&\leq \int_0^\pi \frac{|e^{iaRe^{i\theta}}|}{|R^2e^{2i\theta}+1|}d\theta \\
&\leq \int_0^\pi \frac{e^{-aR\sin\theta}}{R^2-1}d\theta \\
&\leq \int_0^\pi \frac{1}{R^2-1}d\theta \qquad (1) \\
&= \frac{\pi}{R^2-1} \to 0 \qquad R\to\infty
\end{aligned}
$$

With (1) because $\sin\theta \geq 0$ on $[0,\pi]$ so $-aR\sin\theta \leq 0$ on this circle.

Therefore, by the Residue Theorem, we get

$$
\begin{aligned}
2\pi i \text{Res}_{z=i}\frac{e^{iaz}}{z(z^2+1)} &= 2\pi i\frac{e^{iai}}{i(i+i)} \\
&= 2\pi i\frac{e^{-a}}{-2} \\
&= -\pi i e^{-a} \\
&= \lim_{R\to\infty}\lim_{\varepsilon\to 0}(I_1+I_2+I_\varepsilon+I_R) \\
&= -\pi i + \int_\varepsilon^R \frac{2i\sin(ax)}{x(x^2+1)}dx \\
\implies \pi - \pi e^{-a} &= \int_\varepsilon^R \frac{2\sin(ax)}{x(x^2+1)}dx \\
&= \int_{-\infty}^\infty \frac{\sin(ax)}{x(x^2+1)}dx
\end{aligned}
$$

✌

Problem 2. Let $f(z)$ be analytic for $0 < |z| < 1$. Assume there are $C > 0$ and $m \geq 1$ such that

$$|f^{(m)}(z)| \leq \frac{C}{|z|^m}, \qquad 0 < |z| < 1.$$

show that f has a removable singularity at $z = 0$.

Solution. f has a removable singularity if $\lim_{z\to 0} zf(z) = 0$.

Since f is analytic in the open disk, we can write a Laurent expansion for f. Namely,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n = \sum_{n=-\infty}^{-1} \frac{a_n}{z^n} + a_0 + \sum_{n=1}^{\infty} a_n z^n$$

Since

$$f^{(m)}(z) = \sum_{-\infty}^{-1} \frac{(-1)^m((n+m)!a_n}{n!z^{m+n}} + \sum_{n=m}^{\infty} \frac{n!}{(n-m-1)!} a_n z^{n-m-1}.$$

Near 0, the positive powers shrink to nothing. However, for $0 < |z| < \varepsilon$, $z^m f^{(m)}$ is bounded by C and this is not possible for the sum on the left unless the sum on the left is zero.

Namely, $|z^m|\frac{1}{|z^{m+n}|} = \frac{1}{|z|^n}$ which grows arbitrarily large for z near 0.

Therefore, $f(z)$ has no negative powers in its Laurent expansion and so

$$\lim_{z\to 0} zf(z) = \lim_{z\to 0} z \sum_{n=0}^{\infty} a_n z^n = 0.$$

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Problem 3. Let $D \subset \mathbb{C}$ be a connected open set and let (u_n) be a sequence of harmonic functions $u_n : D \to (0, \infty)$. Show that if $u_n(z_0) \to 0$ for some $z_0 \in D$, then $u_n \to 0$ uniformly on compact subsets of D.

Solution. First, we note that u_n are positive. Let $D' \subset D$ be a disk of radius r centered at z_0. Then, for all z with $|z - z_0| < \varepsilon < r$, we can apply Harnack's inequality to get

$$\frac{r-\varepsilon}{r+\varepsilon}u_n(z_0) \leq u_n(z) \leq \frac{r+\varepsilon}{r-\varepsilon}u_n(z_0).$$

By Harnock's inequality, since $u_n(z_0) \to 0$, we must get that $u_n \to 0$ on D'.

Furthermore, the convergence is uniform since for all $\delta > 0$, for N such that $u_n(z_0) < \frac{r-\varepsilon}{r+\varepsilon}\delta$ for all $n \geq N$, we get that

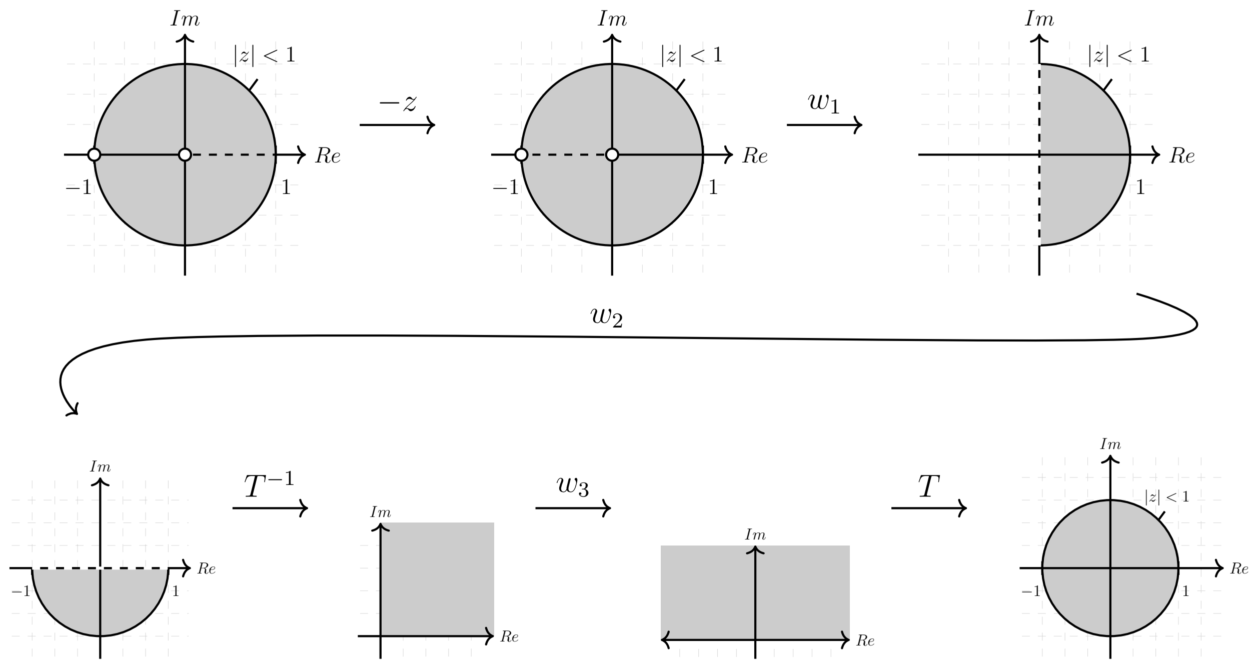
$$u_n(z) \leq \frac{r+\varepsilon}{r-\varepsilon}u_n(z_0) < \frac{r+\varepsilon}{r-\varepsilon}\frac{r-\varepsilon}{r+\varepsilon}\delta = \delta \qquad \text{for all } N \text{ for all } z \in D'.$$

Since we can cover any compact subset of D' with disks, we get that $u_n \to 0$ uniformly on compact subsets. ✌

Problem 4. Let D be the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$ in the complex plane, and define $\Omega = D \backslash [0, 1]$. Find a conformal mapping of Ω onto D. You may give your answer as the composition of several mappings, so long as each mapping is precisely described.

Solution. Let

$$\begin{aligned} T(z) &= \frac{z - i}{z + i} \\ w_1(z) &= z^{\frac{1}{2}} \qquad \text{branch at } (-\infty, 0] \\ w_2(z) &= -iz \\ w_3(z) &= z^2 \end{aligned}$$



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