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Complex Analysis Exam Fall 2012

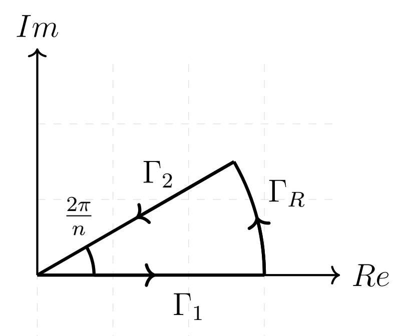
Problem 1. Evaluate the integral

$$\int_0^\infty \frac{dx}{1+x^n}, \qquad n \geq 2$$

being careful to justify your methods.

Solution. Then we get that there is a pole at $e^{i\frac{\pi}{n}}$ which can be isolated in a pizza slice of angle $\frac{2\pi}{n}$.

Thus, we integrate around the following contour:



Immeidately, we get that

$$|I_R| = \left|\int_{\Gamma_R} \frac{dz}{1+z^n}\right| \leq \int_0^{\frac{2\pi}{n}} \frac{R}{R^n-1}d\theta = \frac{2\pi R}{n(R^n-1)} \to 0 \qquad R \to \infty.$$

Since

$$I_2 = \int_{\Gamma_2} \frac{dz}{1+z^n} = \int_R^0 \frac{e^{i\frac{2\pi}{n}}dr}{1+r^n e^{2\pi i}} = -e^{i\frac{2\pi}{n}} \int_0^R \frac{dr}{1+r^n} = -e^{i\frac{2\pi}{n}} I_1.$$

Thus, using the residue theorem, we get that

$$\text{Res}_{z=e^{i\frac{\pi}{n}}} \frac{1}{1+x^n} = \lim_{z\to e^{i\frac{\pi}{n}}} \frac{x-e^{i\frac{\pi}{n}}}{x^n+1} = \lim_{z\to e^{i\frac{\pi}{n}}} \frac{1}{nx^{n-1}} = \frac{1}{n}e^{-(n-1)i\frac{\pi}{n}}$$

and that

$$2\pi i\frac{1}{n}e^{-(n-1)i\frac{\pi}{n}} = \lim_{R\to\infty}(I_1+I_2+I_R) = (1-e^{i\frac{2\pi}{n}})I_1$$

and so

$$\int_0^\infty \frac{dx}{1+x^n} = \frac{\pi}{n} e^{-(n-1)i\frac{\pi}{n}} \frac{2i}{1-e^{i\frac{2\pi}{n}}} = \frac{\pi}{n} \frac{-2i}{e^{-i\frac{\pi}{n}} - e^{i\frac{\pi}{n}}} = \frac{\pi/n}{\sin(\pi/n)}$$

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Problem 2. Find the Laurent series expansion for

$$\frac{1}{z(z+1)}$$

valid in $\{1 < |z-1| < 2\}$.

Solution. Since on $1 < |z-1| < 2$ we have that $1 > \frac{1}{|z-1|} > \frac{1}{2}$ and $\frac{|z-1|}{2} < 1$ so

$$\begin{aligned}
\frac{1}{z(z+1)} &= \frac{1}{z} - \frac{1}{z+1} \\
&= \frac{1}{(z-1)+1} - \frac{1}{(z-1)+2} \\
&= \frac{\frac{1}{z-1}}{1+\frac{1}{z-1}} - \frac{\frac{1}{2}}{1+\frac{z-1}{2}} \\
&= \frac{1}{z-1}\frac{1}{1-\frac{1}{1-z}} - \frac{1}{2}\frac{1}{1-\frac{1-z}{2}} \\
&= \frac{1}{z-1}\sum_{l=0}^{\infty}\left(\frac{1}{1-z}\right)^{l} - \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{1-z}{2}\right)^{k} \\
&= \sum_{l=0}^{\infty}\frac{1}{(1-z)^{l+1}} - \sum_{k=0}^{\infty}\frac{(1-z)^{k}}{2^{k+1}}
\end{aligned}$$

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Problem 3. Suppose that f is an entire function and that there is a bounded sequence of distinct real numbers $a_1, a_2, a_3, ...$ such that $f(a_k)$ is real for each k. Show that $f(x)$ is real for all real x.

Solution. It suffices to show that $f(z)$ has a Taylor Expansion with only real coefficeints.

First, note that since a_k are bounded, they have a real limit point a. Therefore, because f is entire (namely continuous),

$$\lim_{k\to\infty} f(a_k) = f\left(\lim_{k\to\infty} a_k\right) = f(a)$$

and $f(a) \in \mathbb{R}$ since $f(a_k) \in \mathbb{R}$ for all k.

Therefore,

$$\frac{f(a_k) - f(a)}{a_k - a} \in \mathbb{R} \qquad \text{for all } k$$

and so $f'(a) \in \mathbb{R}$.

Inductively, we get that $f^{(n)}(a) \in \mathbb{R}$ for all n and so, because f is entire, we can write its Taylor expansion around a and get that

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(z-a)^k$$

and since $f^{(k)}(a)/k! \in \mathbb{R}$ for all k and $a \in \mathbb{R}$, we get that $f(x) \in \mathbb{R}$ for all $x \in \mathbb{R}$. ✌

Problem 4. Suppose

$$f_n(z) = \sum_{k=0}^{n} \frac{1}{k! z^k}, \qquad z \neq 0$$

and let $\varepsilon > 0$. Show that for large enough n, all the zeros of f_n are in the disk $D(0, \varepsilon)$ with center 0 and radius ε.

Solution. First,

$$\lim_{n\to\infty} f_n(z) = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{z}\right)^k}{k!} = e^{\frac{1}{z}}.$$

Furthermore, this convergence is uniform. Namely, for $\varepsilon > 0$, there exists N such that $|f_n(z) - e^{1/z}| < \varepsilon$ for all $n \geq N$ and all $z \neq 0$.

Now, if $|z| < \frac{1}{\varepsilon}$ then

$$|e^z| \geq e^{-\frac{1}{\varepsilon}}$$

and so if $|z| > \varepsilon$n then

$$|e^{\frac{1}{z}}| \geq e^{-\frac{1}{\varepsilon}} > 0.$$

Namely, because $e^{1/z}$ has a lower bound on $|z| > \varepsilon$, so too must $f_n(z)$. Thus, the zeros of $f_n(z)$ are contained in $D(0, \varepsilon)$. ✌