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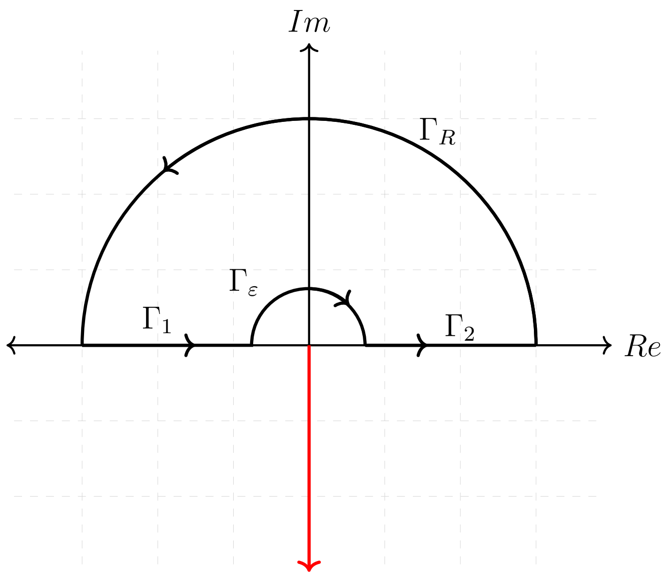
Complex Analysis Exam Spring 2011

Problem 1. Evaluate
$$\int_0^\infty \frac{\log x}{(x^2+1)^2}dx$$

Solution. We will use "Ol' Faithful" the contour around the upper half plane avoiding the origin since every branch cut of $\log x$ intersects 0.

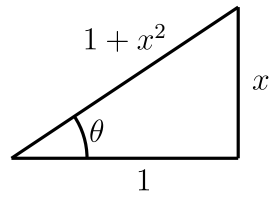
Then we take any branch which does not intersect the upper half plane (including the real line).



Let

$$I_1 = \int_{\Gamma_1} \frac{\log z}{(z^2+1)^2}dz$$
$$I_2 = \int_{\Gamma_2} \frac{\log z}{(z^2+1)^2}dz$$
$$I_\varepsilon = \int_{\Gamma_\varepsilon} \frac{\log z}{(z^2+1)^2}dz$$
$$I_R = \int_{\Gamma_R} \frac{\log z}{(z^2+1)^2}dz$$

For the next computation we refer to the following triangle:



$$
\begin{aligned}
I_1 &= \int_{\Gamma_1} \frac{\log z}{(z^2+1)^2} dz \\
&= \int_{-R}^{-\varepsilon} \frac{\log x}{(x^2+1)^2} dx \\
&= \int_{R}^{\varepsilon} \frac{-\log(-x)}{(x^2+1)^2} dx \\
&= \int_{\varepsilon}^{R} \frac{\log x + i\pi}{(x^2+1)^2} dx \qquad \log(-x) = \log x + i\pi \\
&= I_2 + \int_{\varepsilon}^{R} \frac{i\pi}{(x^2+1)^2} dx \\
&= I_2 + i\pi \int_{\varepsilon}^{R} \frac{\sec^2\theta}{(\tan^2\theta+1)^2} d\theta \qquad x = \tan\theta \\
&= I_2 + i\pi \int_{\varepsilon}^{R} \frac{\sec^2\theta}{\sec^4\theta} d\theta \\
&= I_2 + i\pi \int_{\varepsilon}^{R} \frac{1}{\sec^2\theta} d\theta \\
&= I_2 + i\pi \int_{\varepsilon}^{R} \cos^2\theta d\theta \\
&= I_2 + i\pi \frac{1}{2} \int_{\varepsilon}^{R} 1 + \cos(2\theta) d\theta \\
&= I_2 + \frac{i\pi}{2}\left[\theta + \frac{1}{2}\sin(2\theta)\right]\Big|_{x=\varepsilon}^{R} \\
&= I_2 + \frac{i\pi}{2}\left[\tan^{-1}(x) + \sin(\theta)\cos(\theta)\right]\Big|_{x=\varepsilon}^{R} \\
&= I_2 + \frac{i\pi}{2}\left[\tan^{-1}(x) + \frac{x}{(x^2+1)^2}\right]\Big|_{x=\varepsilon}^{R} \\
&\to I_2 + \frac{i\pi}{2}\left[\frac{\pi}{2}\right] \qquad \text{as } \varepsilon \to 0, R \to \infty \\
&= I_2 + i\frac{\pi^2}{4}
\end{aligned}
$$

Now,

$$\begin{aligned}
|I_R| &= \left|\int_{\Gamma_R} \frac{\log z}{(z^2+1)^2} dz\right| \\
&\leq \int_{\Gamma_R} \frac{|\log z|}{|z^2+1|^2}|dz| \\
&\leq \int_{\Gamma_R} \frac{|z|+C}{|z^2+1|^2} dz \qquad (1) \\
&\leq \int_{\Gamma_R} \frac{R+C}{|z|^2} dz \qquad (2) \\
&= \int_{\Gamma_R} \frac{R+C}{R^2} dz \to 0 \qquad \text{as } R \to \infty
\end{aligned}$$

Note that (1) comes from

$$|\log z| = \sqrt{(\log|z|)^2 + (\arg(z))^2} \leq \sqrt{(\log|z|)^2} + C = \log|z| + C \leq |z| + C$$

for C sufficiently large with respect to the maximal possible argument of z, which is $\frac{3\pi}{2}$.

And (2) comes from $|z^2+1|^2 \geq |z^2+1| \geq |z^2|$ for $|z|$ sufficiently large.

Finally,

$$\begin{aligned}
|I_\varepsilon| &= \left|\int_{\Gamma_\varepsilon} \frac{\log z}{(z^2+1)^2} dz\right| \\
&\leq \int_\pi^0 \frac{|\log(\varepsilon e^{i\theta})||i\varepsilon e^{i\theta}|}{|\varepsilon^2 e^{2i\theta}+1|^2} d\theta \\
&\leq \int_\pi^0 \frac{\varepsilon(|\log\varepsilon| + |\theta|)}{|\varepsilon^2 e^{2i\theta}+1|^2} d\theta \to 0 \qquad \varepsilon \to 0
\end{aligned}$$

Note that

$$\begin{aligned}
\lim_{\varepsilon\to 0} \frac{\varepsilon(|\log\varepsilon| + |\theta|)}{|\varepsilon^2 e^{2i\theta}+1|} &= \left(\lim_{\varepsilon\to 0} \frac{|\log\varepsilon|}{\frac{1}{\varepsilon}}\right)\left(\lim_{\varepsilon\to 0} \frac{1}{|\varepsilon^2 e^{2i\theta}+1|^2}\right) + \lim_{\varepsilon\to 0} \frac{\varepsilon|\theta|}{|\varepsilon^2 e^{2i\theta}+1|^2} \\
&= \lim_{\varepsilon\to 0} \frac{\frac{1}{\varepsilon}}{\frac{-1}{\varepsilon^2}} \cdot 1 + 0 \\
&= \lim_{\varepsilon\to 0} -\varepsilon = 0
\end{aligned}$$

by L'Hopital's rule.

Therefore, by the residue theorem,

$$\begin{aligned}
\lim_{\varepsilon\to 0}\lim_{R\to\infty} I_1+I_2+I_\varepsilon+I_R &= 2\pi i \mathrm{Res}_{z=i}\frac{\log z}{(z^2+1)^2}\\
&= 2\pi i \frac{d}{dz}\frac{\log z}{(z+i)^2}\Big|_{z=i}\\
&= 2\pi i \frac{\frac{(z+i)^2}{z}-2(z+i)\log z}{(z+i)^4}\Big|_{z=i}\\
&= 2\pi i \frac{\frac{(2i)^2}{i}-2(2i)\log i}{(2i)^4}\\
&= 2\pi i \frac{4i-4i\frac{\pi}{2}i}{16}\\
&= -\frac{\pi}{2}\left(1-\frac{\pi}{2}i\right)\\
&= \frac{\pi^2}{4}i-\frac{\pi}{2}\\
\lim_{\varepsilon\to 0}\lim_{R\to\infty} I_1+I_2+I_\varepsilon+I_R &= 2I_1+\frac{\pi^2}{4}i\\
\implies I_1 &= \frac{1}{2}\left(\frac{\pi^2}{4}i-\frac{\pi}{2}-\frac{\pi^2}{4}i\right)\\
&= -\frac{\pi}{4}
\end{aligned}$$

✌

Problem 2.

(a) Suppose that $u_1, u_2, ..., u_n$ and $u_1^2 + \cdots + u_n^2$ are harmonic functions on a connected open set D. Show that each function u_r $(1 \leq r \leq n)$ is constant.

(b) A function $f : D \to \mathbb{C}$ with $f(x+iy) = u(x,y) + iv(x,y)$ is said to be complex harmonic if the real valued functions u and v are harmonic. Show that if $f(x+iy)$ and $(x+y)f(x+iy)$ are both complex harmonic then f is analytic.

TYPO : *This question does not make sense unless we assume that $(x+iy)f(x+iy)$ is complex harmonic.*

Solution.

(a) Note that

$$\begin{aligned}
\frac{\partial}{\partial x}u^2 &= 2u\frac{\partial u}{\partial x} \\
\frac{\partial^2}{\partial x^2}u^2 &= \frac{\partial}{\partial x}\left(2u\frac{\partial u}{\partial x}\right) \\
&= 2\left(\frac{\partial u}{\partial x}\right)^2 + 2u\frac{\partial^2 u}{\partial x^2} \\
\frac{\partial^2}{\partial x^2}u^2 + \frac{\partial^2}{\partial y^2}u^2 &= 2\left(\frac{\partial u}{\partial x}\right)^2 + 2u\frac{\partial^2 u}{\partial x^2} + 2\left(\frac{\partial u}{\partial y}\right)^2 + 2u\frac{\partial^2 u}{\partial y^2} \\
&= 2\left[2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial u}{\partial y}\right)^2\right]
\end{aligned}$$

which is 0 if and only if $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$.

Therefore, since differentiation is a linear operation, $u_1^2 + \cdots + u_n^2$ is harmonic if and only if

$$\frac{\partial u_k}{\partial x} = \frac{\partial u_k}{\partial y} \qquad \text{for all } k.$$

Namely, $u_k = c_k$ for some c_k constant for all k.

(b) We will write $u_x = \frac{\partial}{\partial x}u$.

Original Hypothesis Now, because $(x+y)f(x+iy)$ is complex harmonic,

$$\begin{aligned}
0 &= \frac{\partial^2}{\partial x^2}(x+y)u + \frac{\partial^2}{\partial y^2}(x+y)u \\
&= \frac{\partial}{\partial x}\left[u + (x+y)u_x\right] + \frac{\partial}{\partial y}\left[u + (x+y)u_y\right] \\
&= u_x + u_x + (x+y)u_{xx} + u_y + u_y + (x+y)u_{yy} \\
&= 2u_x + 2u_y
\end{aligned}$$

since $f(x+iy)$ is complex harmonic so $u_{xx} = -u_{yy}$.

Therefore, $u_x = -u_y$ and similarly, $v_x = -v_y$.

However, to obtain the final solution, it is necessary to assume that $(x+iy)f(x+iy)$ is complex harmonic, instead of $(x+y)f(x+iy)$.

$\boxed{\text{Assuming } (x+iy)f(x+iy) \text{ is complex harmonic}}$ Now,

$$(x+iy)f(x+iy) = (x+iy)(u+iv) = xu - yv + i(yu + xv)$$

and so the real part being harmonic now implies that

$$\begin{aligned} 0 &= \frac{\partial^2}{\partial x^2}(xu - yv) + \frac{\partial^2}{\partial y^2}(xu - yv) \\ &= \frac{\partial}{\partial x}\left[u + xu_x - yv_x\right] + \frac{\partial}{\partial y}\left[xu_y - v - yv_y\right] \\ &= u_x + u_x + xu_{xx} - yv_{xx} + xu_{yy} - v_y - v_y - yv_{yy} \\ &= 2u_x - 2v_y \end{aligned}$$

since $f(x+iy)$ is complex harmonic so $xu_{xx} + xu_{yy} = 0$ and similarly, $-yv_{xx} - yv_{yy} = 0$.

Now finally, the complex part of $f(x+iy)$ is harmonic and so

$$\begin{aligned} 0 &= \frac{\partial^2}{\partial x^2}(yu + xv) + \frac{\partial^2}{\partial y^2}(yu + xv) \\ &= \frac{\partial}{\partial x}\left[yu_x + v + xv_x\right] + \frac{\partial}{\partial y}\left[u + yu_y + xv_y\right] \\ &= yu_{xx} + v_x + v_x + xv_{xx} + u_y + u_y + yu_{yy} + xv_{yy} \\ &= 2v_x + 2u_y \end{aligned}$$

and so $u_x = v_y$ and $u_y = -v_x$. Therefore, by Cauchy-Riemann, $f(x+iy)$ is analytic.

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Problem 3. Let $f : D \to D$ be an analytic function on a bounded domain D with $0 \in D$. Assume $f(0) = 0$ and $|f'(0)| < 1$. Let $F_n(z) = f \circ \cdots \circ f(z)$ (n-times). Show that $F_n(z) \to 0$ as $n \to \infty$ uniformly on compact subsets of D. Hint: consider first the behavior of F_n on a small neighborhood of 0.

Solution. Since

$$\left|\lim_{z\to 0} \frac{f(z) - f(0)}{z - 0}\right| = \lim_{z\to 0} \frac{|f(z)|}{|z|} = |f'(0)|,$$

we have that for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon > \left|\frac{f(z)}{z} - f'(0)\right| \geq \frac{|f(z)}{|z|} - |f'(0)| \qquad |z| < \delta.$$

Therefore,

$$\frac{|f(z)|}{|z|} < \varepsilon + |f'(0)|.$$

Since $\varepsilon > 0$ was arbitrary and $|f'(0)| < 1$, there exists $0 < \rho < 1$ such that

$$\frac{|f(z)|}{|z|} \leq \rho < 1$$

for all $|z| < \delta$.

Note, that since D is the domain of an analytic function we may take it to be connected and open, so we can take δ such that $B_\delta(0) \subset D$. Namely, we have that $|f(z)| \leq \rho|z|$ for all $z \in B_\delta(0) \subset D$.

Finally, since $\rho < 1$, $\rho z \in B_\delta(0)$ for all $z \in B_\delta(0)$.

Thus,

$$|f\circ\cdots\circ f(z)| \text{ (n-times) } \leq |f\circ\cdots\circ f(\rho z)| \text{ ($n{-}1$-times) } = |f\circ\cdots\circ f(\rho^2 z) \text{ ($n{-}2$-times) } \leq \cdots \leq \rho^n|z|.$$

Therefore,

$$\lim_{n\to\infty} |f \circ \cdots \circ f(z)| \text{ (n-times) } \leq \lim_{n\to\infty} \rho^n|z| = 0$$

for all $z \in B_\delta(0)$ since $\rho < 1$.

Next, assume that $K \subset D$ is compact. By Cauchy, because D is bounded, and $f : D \to D$, there exists M such that $|f(z)| \leq M$ for all $z \in D$.

Namely, $|F_n(z)| \leq M$ for all $z \in D$. Therefore, by Montel's Theorem, $\{F_n\}$ define a normal family on D.

Therefore, for every $z \in D$, there exists a subsequence $\{F_{n_k}\}$ of $\{F_n\}$ such that $F_{n_k} \to F$ uniformly on compact subsets $K \subset D$, for some analytic function $F : D \to D$.

However, $F_{n_k}|_{B_\delta(0)} \to 0$ uniformly, which implies that $F|_{B_\delta(0)} = 0$. Since F is analytic, its zeros must be isolated, so $F = 0$ for every z in a ball implies that $F \equiv 0$ is identically 0 on D.

Namely, for each $K \subset D$ compact, there exists a subsequence $\{F_{n_k}\}$ which converges uniformly to 0 on K.

Finally, to show that the whole sequence converges to 0 on K, we note that for all $\varepsilon > 0$, there exists N such that

$$|F_{n_k}| < \varepsilon \qquad \text{for all } n_k \geq N \text{ in the subsequence.}$$

However, from the above, if $|F_{n_k}(z)| < \varepsilon$, then

$$|f(F_{n_k}(z))| \leq \rho|F_{n_k}(z)| < \rho\varepsilon < \varepsilon \qquad \rho < 1$$

and so, since $f \circ F_{n_k}(z) = F_{n_k+1}(z)$, we have that $|F_n(z)| < \varepsilon$ for all $n \geq N$.

Namely, $F_n \to 0$ uniformly on K ✌

Problem 4. Starting with the definition

"f is analytic on a set G if $\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists for all $z_0 \in G$."

describe the sequence of intermediate results required to obtain the following theorem:

"Suppose f and g are both analytic on a connected open set G and there is a convergent sequence z_n with limit $z_\infty \in G$ such that $f(z_n) = g(z_n)$ for all n. Then $f = g$ on G."

You do not need to prove any of the intermediate results, but you should give a brief indication of how each result is used to obtain the next one.

Solution.

THIS QUESTION IS TERRIBLE. Proofs will be provided for the sake of learning...

1. Since f and g are analytic, $f - g$ is analytic by Cauchy-Riemann.
2. The zeros of analytic functions are isolated. Although we used this freely in **Problem 3**, for the sake of understanding we prove this as a claim.

Claim 1. The zeros of an analytic function are isolated.

Proof. Let h be an analytic function and let z_0 be a zero of h. Then, because h is analytic, we can develop its Taylor series about z_0 in $B_R(z_0)$ for some $R > 0$. Namely,

$$h(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n.$$

Now, since z_0 is a zero of h, $a_0 = 0$, and so WLOG, we can take N to be the largest integer such that $a_n = 0$ for all $0 \leq n < N$ and $a_N \neq 0$. Then,

$$h(z) = \sum_{n=N}^{\infty} a_n(z-z_0)^n = \sum_{n=0}^{\infty} a_{n+N}(z-z_0)^{n+N} = (z-z_0)^N \sum_{n=0}^{\infty} a_{n+N}(z-z_0)^n$$

Therefore, $h(z) = (z - z_0)^N k(z)$ with $k(z) = \sum_{n=0}^{\infty} a_{n+N}(z - z_0)^n$ also an analytic function on $B_R(z_0)$. Furthermore, $k(z_0) = a_N \neq 0$ by assumption.

Thus, because k is analytic, it is continuous, and because $k(z_0)$ non-zero, there exists a $\delta > 0$ such that

$$|k(z) - a_N| < \frac{|a_N|}{2} \qquad |z - z_0| < \delta$$

> and so $k(z) \not 0$ on $B_\delta(z_0)$.
>
> Namely, z_0 must be an isolated singularity of h. ✌

3. Because $f - g$ is analytic (by 1.) and $f - g = 0$ on a sequence in G, the zeros of $f - g$ cannot be isolated. This is because limit points are not isolated by definition.

   Therefore, assuming that $f - g$ is not identically 0 contradicts 2. and so $f - g \equiv 0$ on G. Thus, $f = g$ on G.

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