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Complex Analysis Exam Fall 2011

Problem 1. Evaluate
$$\int_0^{2\pi} \frac{d\theta}{3+\cos\theta+2\sin\theta}.$$

Solution.
$$\begin{aligned}
\int_0^{2\pi} \frac{d\theta}{3+\cos\theta+2\sin\theta} &= \int_0^{2\pi} \frac{d\theta}{3+\frac{e^{i\theta}+e^{-i\theta}}{2}+\frac{e^{i\theta}-e^{-i\theta}}{i}} \\
&= \int_0^{2\pi} \frac{2ie^{i\theta}d\theta}{6ie^{i\theta}+ie^{2i\theta}+i+2e^{2i\theta}-2} \\
&= \int_{|z|=1} \frac{2dz}{6iz+iz^2+i+2z^2-2} \qquad z=e^{i\theta} \\
&= \int_{|z|=1} \frac{2dz}{(i+2)z^2+6iz+i-2} \\
&= \int_{|z|=1} \frac{2dz}{(i+2)\left(z+\frac{1}{5}(1+2i)\right)(z+1+2i)} \qquad (1) \\
&= \left(2\pi i \text{Res}_{z=-\frac{1}{5}(1+2i)} \frac{2}{(i+2)\left(z+\frac{1}{5}(1+2i)\right)(z+1+2i)}\right) \\
&= 2\pi i \frac{2}{(i+2)\left(-\frac{1}{5}(1+2i)+1+2i\right)} \\
&= 4\pi i \frac{1}{(i+2)\frac{4}{5}(1+2i)} \\
&= 5\pi i \frac{1}{(1+2i)(i+2)} \\
&= 5\pi i \frac{1}{i+2-2+4i} \\
&= 5\pi i \frac{1}{5i} \\
&= \pi
\end{aligned}$$

With (1) from the quadratic formula where

$$\begin{aligned} z &= \frac{-6i \pm \sqrt{-36 - 4(i+2)(i-2)}}{2(i+2)} \\ &= \frac{-6i \pm \sqrt{-36 - 4(-5)}}{2(i+2)} \\ &= \frac{-6i \pm \sqrt{-16}}{2(i+2)} \\ &= \frac{-6i \pm 4i}{2(i+2)} \\ &= \frac{-3i \pm 2i}{i+2} \\ &= \frac{-i}{i+2}, \frac{-5i}{i+2} \\ &= -\frac{1}{5}(-i(i-2)), i(i-2) \\ &= -\frac{1}{5}(1+2i), -1-2i \end{aligned}$$

and since $|-1-2i| > 1$ and $\frac{1}{5}|1+2i| < 1$, we have only one residue. ✌

Problem 2. Suppose the series $f(z) = \sum_{n=0}^{\infty} c_n z^n$ converges for $|z| < R$. Show that for $r < R$,

$$\int_{|z|=r} |f(z)|^2 dz = 2\pi \sum_{n=0}^{\infty} |c_n|^2 r^{2n}.$$

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$$\int_{|z|=r} |f(z)|^2 |dz| = 2\pi r \sum_{n=0}^{\infty} |c_n|^2 r^{2n}.$$

Solution. We must interpret

$$\int_{|z|=r} |f(z)|^2 dz = \int_{|z|=r} |f(z)|^2 |dz|$$

else this problem does not make sense. Specifically, if $f(z) = z$, which certainly has a well defined Taylor series ($c_n = 0$ for all $n \neq 1$), then

$$\int_{|z|=r} |f(z)|^2 dz = \int_{|z|=r} |z|^2 dz = r^2 \int_{|z|=r} dz = 0 \neq 2\pi r^2.$$

Now, with this change in notation, we obtain that

$$\int_{|z|=r} |f(z)|^2 |dz| = \int_{|z|=r} |z|^2 |dz| = r^2 \int_{|z|=r} |dz| = 2\pi r^3 = 2\pi r(r^2) = 2\pi r \sum_{n=0}^{\infty} |c_n|^2 r^{2n}.$$

$$\begin{aligned}
\int_{|z|=r} |f(z)|^2 |dz| &= \int_{|z|=r} f(z)\overline{f(z)} |dz| \\
&= \int_{|z|=r} \left(\sum_{n=0}^{\infty} c_n z^n\right) \left(\overline{\sum_{n=0}^{\infty} c_n z^n}\right) |dz| \\
&= \int_{|z|=r} \left(\sum_{n=0}^{\infty} c_n z^n\right) \left(\sum_{n=0}^{\infty} \overline{c_n z}^n\right) |dz| \\
&= \int_{|z|=r} \sum_{n=0}^{\infty} \sum_{k=0}^{n} c_k z^k \overline{c_{n-k} z}^{n-k} |dz| \qquad \text{Cauchy Product} \\
&= \sum_{n=0}^{\infty} \int_{|z|=r} \sum_{k=0}^{n} c_k \overline{c_{n-k}} z^k \overline{z}^{n-k} |dz|
\end{aligned}$$

Now, we examine the inner sum.

Claim 1.

$$\int_{|z|=r} z^n |dz| = 0 \qquad \text{for all } n \neq 0, n \in \mathbb{N}.$$

Proof. Assume $n \neq 0$. Then let $z = re^{i\theta}$. Then $dz = ire^{i\theta} d\theta$ so $|dz| = r d\theta$.

Therefore,

$$\begin{aligned}\int_{|z|=r} z^n|dz| &= \int_0^{2\pi}(re^{i\theta})^n r d\theta \\ &= \int_0^{2\pi} r^{n+1}e^{ni\theta}d\theta \\ &= r^{n+1}\frac{e^{ni\theta}}{ni}\Big|_0^{2\pi} \\ &= \frac{r^{n+1}}{ni}(e^{2ni\pi}-1) \\ &= \frac{r^{n+1}}{ni}(1-1) \\ &= 0\end{aligned}$$

since n is an integer, $e^{2ni\pi} = \cos(2n\pi) + i\sin(2n\pi) = 1$.

Note that if $n = 0$, then

$$\int_{|z|=r} z^n|dz| = \int_0^{2\pi} r d\theta = 2\pi r.$$

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$$\sum_{k=0}^{n} c_k\overline{c_{n-k}}z^k\overline{z}^{n-k} = \begin{cases}\sum_{0\le j<k\le n} c_j\overline{c_k}z^j\overline{z}^k + \sum_{n\ge j>k\ge 0} c_j\overline{c_k}z^j\overline{z}^k & \text{if } n \text{ is odd} \\ \sum_{0\le j<k\le n} c_j\overline{c_k}z^j\overline{z}^k + \sum_{n\ge j>k\ge 0} c_j\overline{c_k}z^j\overline{z}^k + c_{n/2}\overline{c_{n/2}}z^{n/2}\overline{z^{n/2}} & \text{if } n \text{ is even}\end{cases}$$

Now, we simply cite the claim.

If $j < k$ then

$$c_j\overline{c_k}z^j\overline{z}^k = C|z|^k\frac{1}{z^{k-j}} \qquad k-j>0$$

and so these integrals will die since they contain a z^l term with $l \neq 0$.

Similarly, if $j > k$ then

$$c_j\overline{c_k}z^j\overline{z}^k = C|z|^k z^{j-k} \qquad j-k>0$$

so these integrals will also die.

Therefore, if n is odd, all terms die.

If n is even, then the only term which will not contain a power of z is in fact

$$c_{n/2}\overline{c_{n/2}}z^{n/2}\overline{z^{n/2}} = |c_n|^2|z^n|^2 = |c_n|^2|z|^{2n} \qquad \text{after reindexing.}$$

In this case,

$$\int_{|z|=r}|c_n|^2|z|^{2n}dz = |c_n|^2r^{2n}\int_{|z|=r}|dz| = |c_n|^2r^{2n+1}2\pi.$$

Finally, we have that

$$\int_{|z|=r} |f(z)|^2 |dz| = \sum_{n=0}^{\infty} \int_{|z|=r} \sum_{k=0}^{n} c_k \overline{c_{n-k}} z^k \overline{z}^{n-k} |dz| = \sum_{n=0}^{\infty} |c_n|^2 r^{2n} \int_{|z|=r} |dz| = 2\pi r \sum_{n=0}^{\infty} |c_n|^2 r^{2n}.$$

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Problem 3. Let $f(z)$ be analytic on $\mathbb{C}$ and suppose that the line $\Gamma = \{t + it \,|\, t \in \mathbb{R}\}$ is mapped to itself, that is $f(z) \in \Gamma$ for all $z \in \Gamma$. If $f(\sqrt{2}) = 3$, then what is $f(\sqrt{2}i)$

Solution. Let

$$g(z) = e^{-\frac{\pi}{4}i}z = \frac{\sqrt{2}}{2}(1-i)z.$$

Note that g is trivially a Mobius transform and so it is invertible.

Then

$$g(t+it) = e^{-\frac{\pi}{4}i}(t+it) = \frac{\sqrt{2}}{2}t(1-i)(1+i) = \frac{\sqrt{2}}{2}t2 = \sqrt{2}t.$$

Thus, g sends Γ to the real line.

Thus, $g \circ f \circ g^{-1}$ fixes the real line. Since g and g^{-1} are analytic (clearly) and f is analytic, their composition is analytic.

Therefore, by Schwarz' Reflection Principle,

$$(g \circ f \circ g^{-1})(z) = \overline{(g \circ f \circ g^{-1})(\overline{z})}.$$

Now, $g(\sqrt{2}) = 1 - i$ and $g(\sqrt{2}i) = 1 + i = \overline{1-i}$ so

$$\begin{aligned}
(g \circ f \circ g^{-1})(1-i) &= g(f(\sqrt{2})) \\
&= g(3) \\
&= \frac{3\sqrt{2}}{2}(1-i) \\
&= \overline{(g \circ f \circ g^{-1})(\overline{1-i})} \\
&= \overline{g(f(g^{-1}(1+i)))} \\
&= \overline{g(f(\sqrt{2}i))} \\
g(f(\sqrt{2}i)) &= \overline{\frac{3\sqrt{2}}{2}(1-i)} \\
&= \frac{3\sqrt{2}}{2}(1+i) \\
f(\sqrt{2}i) &= g^{-1}\left(\frac{3\sqrt{2}}{2}(1+i)\right) \\
&= g^{-1}\left(\frac{\sqrt{2}}{2}(1-i)(3i)\right) \qquad (1) \\
&= 3i
\end{aligned}$$

with (1) since

$$\frac{3\sqrt{2}}{2}(1+i) = \frac{\sqrt{2}}{2}(1-i)z \implies z = 3\frac{1+i}{1-i} = 3\frac{(1+i)^2}{2} = \frac{3}{2}(1-1+2i) = 3i.$$

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Problem 4. Let $\Omega \subset \mathbb{C}$, with $\Omega \neq \mathbb{C}$, be simply connected, and let $f : \Omega \to \Omega$ be a conformal bijection. If f has two distinct fixed points z_1, z_2, (that is, $f(z_1) = z_1$, $f(z_2) = z_2$), show that f is the identity map.

Solution. Since $\omega \neq \mathbb{C}$ and $\omega \subset \mathbb{C}$, we have that $\omega \subset \overline{\mathbb{C}}$ with at least two points in its compliment (namely, some point in $\mathbb{C}$ and ∞).

Therefore, by the Riemann Mapping Theorem, since Ω is simply connected, there exists an analytic bijection $g : \omega \to \mathbb{D}$ from ω to the unit disk such that $g(z_1) = 0$ and $g'(z_1) \in \mathbb{R}^+$.

Therefore,

$$g \circ f \circ g^{-1} : \mathbb{D} \to \mathbb{D}$$

is a map from the disk to the disk and

$$(g \circ f \circ g^{-1})(0) = g(f(z_1)) = g(z_1) = 0.$$

Furthermore, if $g(z_2) = z$ then

$$(g \circ f \circ g^{-1})(z) = g(f(z_2)) = g(z_2) = z$$

and so by Schwarz' Lemma, $g \circ f \circ g^{-1} = cz$ is a rotation for some $|c| = 1$.

However, clearly $c = 1$ since we have already shown that $z \mapsto z$ through $g \circ f \circ g^{-1}$.

Therefore,

$$f(z) = g^{-1}(g(z)) = z$$

and so f is the identity map. ✌