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Complex Analysis Exam Spring 2010

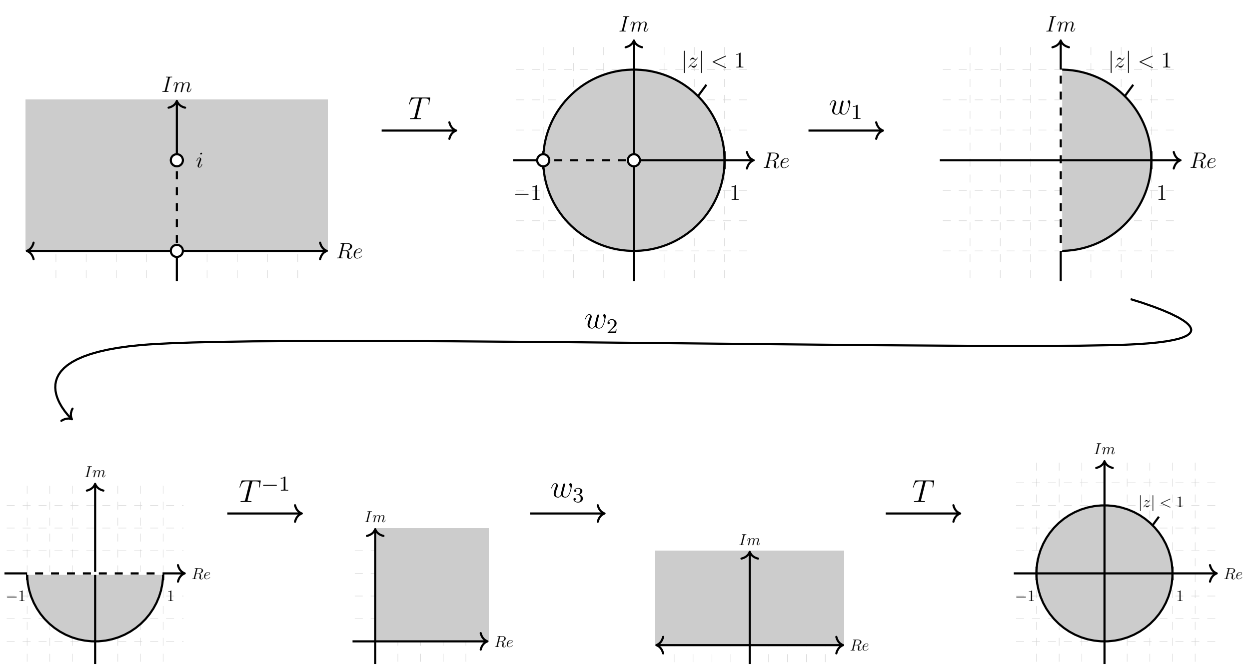
> **Problem 1.** Map the region $\Omega = \{\Im(z) > 0\} \backslash \{iy \,|\, 0 < y \leq 1\}$ conformally to the unit disk $D = \{|z| < 1\}$.

Solution. The issue is that Ω is missing a line segment. To remove this, we must shrink and expand the region to fill in the missing line segment.

Let

$$\begin{aligned} T(z) &= \frac{z-i}{z+i} \\ w_1(z) &= z^{\frac{1}{2}} \qquad \text{branch at } (-\infty, 0] \\ w_2(z) &= -iz \\ w_3(z) &= z^2 \end{aligned}$$

Then,



Problem 2. How many zeros of $p(z) = z^4 + z^3 + 4z^2 + 2z + 7$ lie in the right half plane $\{\mathcal{R}\text{e}(z) > 0\}$?

Solution. First, we check how many times p wraps the imaginary axis (the boundary of the region in question) around the origin.

$$\begin{aligned} p(iy) &= (iy)^4 + (iy)^3 + 4(iy)^2 + 2(iy) + 7 \\ &= y^4 - iy^3 - 4y^2 + 2iy + 7 \\ &= y^4 - 4y^2 + 7 + i(2y - y^3) \end{aligned}$$

Since $\mathcal{R}\text{e}(p(iy))' = 4y^3 - 8y$, we can check that $\mathcal{R}\text{e}(p(iy))$ has minimums at $\pm\sqrt{2}$. Since $y \in \mathbb{R}$ and $\mathcal{R}\text{e}(p(i\sqrt{2})) = 4 - 8 + 7 > 0$ we have that $p(iy)$ always has positive real part and so does not wrap around the origin.

Now, we check how many times p wraps a large circle around the origin. Let $z = Re^{i\theta}$ for $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

Then $\arg(p(Re^{i\theta})) = \arg(\frac{1}{R^4}p(Re^{i\theta}))$ since multiplying by positive real constants does not change the argument.

However,

$$\lim_{R\to\infty} \frac{1}{R^4}p(Re^{i\theta}) = \lim_{R\to\infty} e^{4i\theta} + \frac{e^{3i\theta}}{R} + \frac{4e^{2i\theta}}{R^2} + \frac{2e^{i\theta}}{R^3} + \frac{7}{R^4} = e^{4i\theta}.$$

Since $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $4\theta \in [-2\pi, 2\pi]$. Namely, the total change in argument of $p(z)$ is 4π which implies that $p(Re^{i\theta})$ wraps very large circles around the origin twice.

Therefore, there are $\boxed{2}$ zeros of p in the right-half plane. ✌

Problem 3. Let f be analytic in the unit disk $D = \{|z| < 1\}$ and continuous on its closure $\overline{D}$. Show that if f is real valued on the boundary $\partial D = \{|z| = 1\}$ then f must be constant.

Solution. Write $f = u + iv$. Since v is harmonic, the maximum and minimum principle states that v must attain both its maximum and minimum on the boundary of any simply connected set.

Therefore, since $f(z) \in \mathbb{R}$ for all $|z| = 1$, v is identically 0 on the boundary of the $\mathbb{D}$ the unit disk.

Thus, v is identically 0.

Therefore, $f(z) = u(z) \in \mathbb{R}$.

Since f is analytic, Cauchy-Riemann states that

$$u_x = v_y = 0 \qquad \text{and} \qquad u_y = -v_x = 0$$

so $f(z) = u(z) = c$ for some real constant c. ✌

Problem 4. By consideration of $\int e^{z+\frac{1}{z}}dz$, or otherwise, show that

$$\frac{1}{2\pi}\int_0^{2\pi} e^{2\cos\theta}\cos\theta d\theta = 1 + \frac{1}{2!} + \frac{1}{2!3!} + \frac{1}{3!4!} + \cdots$$

Solution. Using the hint, we note that

$$\int_{|z|=1} e^{z+\frac{1}{z}}dz = \int_0^{2\pi} e^{e^{i\theta}+e^{-i\theta}} ie^{i\theta}d\theta = \int_0^{2\pi} e^{2\cos\theta} i(\cos\theta + i\sin\theta)d\theta.$$

Therefore,

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} e^{2\cos\theta}\cos\theta d\theta &= \frac{1}{2\pi i}\int_{|z|=1} e^{z+1/z}dz - \frac{1}{2\pi}\int_0^{2\pi} e^{2\cos\theta}\sin\theta d\theta \\ &= \frac{1}{2\pi i}\int_{|z|=1} e^{z+1/z}dz + \frac{1}{2\pi}\int_1^1 e^{2u}du \qquad u = \cos\theta \\ &= \frac{1}{2\pi i}\int_{|z|=1} e^{z+1/z}dz \\ &= \frac{1}{2\pi i}\int_{|z|=1} \sum_{n=0}^{\infty}\frac{\left(z+\frac{1}{z}\right)^n}{n!}dz \\ &= \frac{1}{2\pi i}\int_{|z|=1} \sum_{n=0}^{\infty}\frac{(z^2+1)^n}{z^n n!}dz \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\frac{1}{2\pi i}\int_{|z|=1}\frac{(z^2+1)^n}{z^n}dz \\ &= \sum_{n=1}^{\infty}\frac{1}{n!}\frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}(z^2+1)^n\Big|_{z=0}\end{aligned}$$

At this point, it is merely a matter of plugging in the first 8 values of n to obtain the final answer.

It is helpful to note that

$$(z^2+1)^n = \sum_{k=0}^{n}\binom{n}{k}(z^2)^k 1^{n-k} = \sum_{k=0}^{n}\frac{n!}{k!(n-k)!}z^2k.$$

Thus,

$\boxed{n=1}$ $1 + z^2\Big|_{z=0} = 1$

$\boxed{n=2}$ $1 + 2z^2 + z^4$ so the first derivative is 0 at 0.

and so forth... ✌