COMPLEX ANALYSIS GRADUATE EXAM

Spring 2011

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

Residue Integral

1. Evaluate

$$\int_0^\infty \frac{\log x}{(x^2+1)^2}\,dx.$$

2. (i) Suppose that $u_1, u_2, \ldots, u_n$ and $u_1^2 + \cdots + u_n^2$ are harmonic functions on a connected open set D. Show that each function u_r $(1 \leq r \leq n)$ is constant.

(ii) A function $f : D \to \mathbb{C}$ with $f(x+iy) = u(x,y) + iv(x,y)$ is said to be complex harmonic if the real valued functions u and v are harmonic. Show that if $f(x+iy)$ and $(x+iy)f(x+iy)$ are both complex harmonic then f is analytic. TYPO

3. Let $f : D \to D$ be an analytic function on a bounded domain D with $0 \in D$. Assume $f(0) = 0$ and $|f'(0)| < 1$. Let $F_n(z) = f \circ \cdots \circ f(z)$ (n times). Show that $F_n(z) \to 0$ as $n \to \infty$ uniformly on compact subsets of D.

[Hint: consider first the behavior of F_n on a small neighborhood of 0.]

4. Starting with the definition

"f is analytic on a set G if $\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists for all $z_0 \in G$."

describe the sequence of intermediate results required to obtain the following theorem:

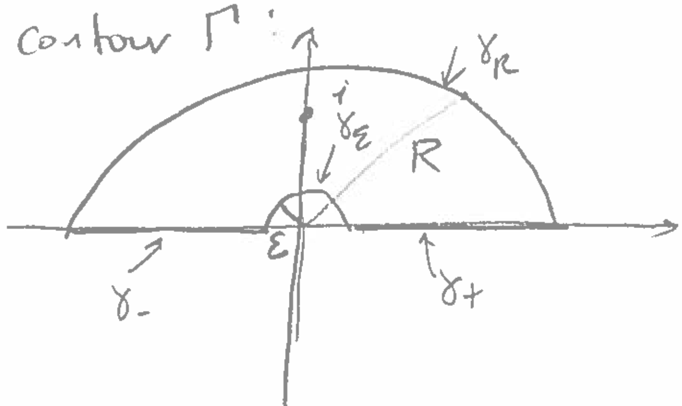
"Suppose f and g are both analytic on a connected open set G and there is a convergent sequence z_n with limit $z_\infty \in G$ such that $f(z_n) = g(z_n)$ for all n. Then $f = g$ on G."

[You do not need to prove any of the intermediate results, but you should give a brief indication of how each result is used to obtain the next one.]

CA - Spring '11

(1.) $\int_0^\infty \frac{\log x}{(x^2+1)^2}\,dx$

Consider the function $f(z) = \frac{\log z}{(z^2+1)^2}$ where we define $\log$ on $\mathbb{C}\setminus\{\text{neg imag. axis}\}$. Then consider the integral over contour Γ:



Here we are letting $\log z = \log|z| + i\arg z$ where $\arg z \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$

- Now consider $\int_{\gamma_R} f(z)\,dz$:

$\arg z \in (0,\pi)$ for $z\in\gamma_R$, $z\in\gamma_\varepsilon$

$$\left|\int_{\gamma_R} \frac{\log z}{(z^2+1)^2}dz\right| \le \int_{\gamma_R}\frac{|\log z|}{|z^2+1|^2}|dz| \le \int_{\gamma_R}\frac{|\log z|}{(R^2-1)^2}|dz|$$

$$= \int_{\gamma_R}\frac{|\log|z| + i\arg z|}{(R^2-1)^2}|dz| \le \int_{\gamma_R}\frac{\log R + \pi}{(R^2-1)^2}|dz| = \frac{\log R + \pi}{(R^2-1)^2}\pi R \to 0$$

as $R\to\infty$

- On the other hand, consider $\int_{\gamma_\varepsilon} f(z)\,dz$:

$$\left|\int_{\gamma_\varepsilon}\frac{\log z}{(z^2+1)^2}dz\right| \le \int_{\gamma_\varepsilon}\frac{|\log|z| + i\arg z|}{(\varepsilon^2-1)^2} \le \int_{\gamma_\varepsilon}\frac{\log\varepsilon + \pi}{(\varepsilon^2-1)^2}$$

$$= \frac{(\log\varepsilon + \pi)}{(\varepsilon^2-1)^2}\pi\varepsilon = \frac{\pi\varepsilon\log\varepsilon + \pi^2\varepsilon}{(\varepsilon^2-1)^2} = \frac{\pi\varepsilon\log\varepsilon}{(\varepsilon^2-1)^2} + \frac{\pi^2\varepsilon}{(\varepsilon^2-1)^2}$$

clearly $\to 0$ as $\varepsilon\to 0$

See that $\lim_{\varepsilon\to0}\varepsilon\log\varepsilon = \lim_{\varepsilon\to0}\frac{\log\varepsilon}{\frac{1}{\varepsilon}}$

L'Hopital

$$\overset{\text{L'H}}{=} \lim_{\varepsilon\to0}\frac{\frac{1}{\varepsilon}}{-\frac{1}{\varepsilon^2}} = \lim_{\varepsilon\to0}\frac{-\varepsilon^2}{\varepsilon} = \lim_{\varepsilon\to0}-\varepsilon = 0$$

hence $\frac{\pi\varepsilon\log\varepsilon}{(\varepsilon^2-1)^2} \to 0$ as $\varepsilon\to0$

Therefore $\int_\Gamma = \int_{\gamma_-} + \int_{\gamma_+}$ as $R \to \infty$ and $\varepsilon \to 0$

Now consider the remaining integrals:

- $$\int_{\gamma_-} \frac{\log z}{(z^2+1)}\,dz = \int_{-R}^{-\varepsilon} \frac{\log|x| + i\pi}{(x^2+1)^2}\,dx$$ since $x \in$ negative real axis

$$= \int_{-R}^{-\varepsilon} \frac{\log(-x) + i\pi}{(x^2+1)^2}\,dx$$

$$= \int_{\varepsilon}^{R} \frac{\log(x) + i\pi}{(x^2+1)^2}\,dx$$

$$= \int_{\varepsilon}^{R} \frac{\log(x)}{(x^2+1)^2}\,dx + \int_{\varepsilon}^{R} \frac{i\pi}{(x^2+1)^2}\,dx$$

and

pos. real axis, so arg = 0

- $$\int_{\gamma_+} \frac{\log z}{(z^2+1)}\,dz = \int_{\varepsilon}^{R} \frac{\log x}{(x^2+1)^2}\,dx$$

Hence as $R \to \infty$, $\varepsilon \to 0$,

$$\int_\Gamma = 2\int_0^\infty \frac{\log x}{(x^2+1)^2}\,dx + i\int_0^\infty \frac{\pi}{(x^2+1)^2}\,dx$$

pole order 2

Now we'll calculate the residue: $$\operatorname{Res}_{z=i} \frac{\log z}{(z^2+1)^2} = \lim_{z\to i} \frac{d}{dz}\left(\frac{(z-i)^2 \log z}{(z^2+1)^2}\right)$$

$$= \lim_{z\to i} \frac{d}{dz}\left(\frac{\log z}{(z+i)^2}\right) = \lim_{z\to i}\left(\frac{1}{z(z+i)^2} + \frac{-2\log z}{(z+i)^3}\right)$$

$$= \frac{1}{i(i+i)^2} + \frac{-2\log i}{(i+i)^3} = \frac{1}{-4i} + \frac{-\cancel{2}(\log 1 + i\frac{\pi}{2})}{-\cancel{8}i} = \frac{1 - i\frac{\pi}{2}}{-4i} = \frac{i\frac{\pi}{2} - 1}{4i}$$

$$= \frac{-\left(-\frac{\pi}{2} - i\right)}{4} = \frac{\frac{\pi}{2} + i}{4} = \frac{\pi + 2i}{8}$$

So: $$\cancel{2}\pi i\left(\frac{\pi + 2i}{4}\right) = 2\int_0^\infty \frac{\log x}{(x^2+1)^2}\,dx + i\int_0^\infty \frac{\pi}{(x^2+1)^2}\,dx$$

$$\frac{\pi^2 i - 2\pi}{4} \xrightarrow{\text{take real part}} 2\int_0^\infty \frac{\log x}{(x^2+1)^2}\,dx = \frac{-2\pi}{4}$$

$$\Rightarrow \int_0^\infty \frac{\log x}{(x^2+1)^2}\,dx = \frac{-\pi}{4}$$

② (i) $u_1, u_2, \dots, u_n$, $u_1^2 + \dots + u_n^2$ are all harmonic on connected, open D.
Show that they are all constant

$$\frac{\partial^2}{\partial x^2}(u_1^2 + \dots + u_n^2) = \frac{\partial}{\partial x}(2u_1 u_{1x} + \dots + 2u_n u_{nx})$$

$$= 2 \frac{\partial}{\partial x}(u_1 u_{1x} + \dots + u_n u_{nx})$$

$$= 2(u_{1x} u_{1x} + u_1 u_{1xx} + u_{2x} u_{2x} + u_2 u_{2xx} + \dots)$$

$$= 2(u_{1x}^2 + \dots + u_{nx}^2 + u_1 u_{1xx} + \dots + u_n u_{nxx})$$

and likewise:

$$\frac{\partial^2}{\partial y^2}(u_1^2 + \dots + u_n^2) = 2(u_{1y}^2 + \dots + u_{ny}^2 + u_1 u_{1yy} + \dots + u_n u_{nyy})$$

$$\Rightarrow 0 = (u_{1x}^2 + \dots + u_{nx}^2 + u_1 u_{1xx} + \dots + u_n u_{nxx}) + (u_{1y}^2 + \dots + u_{ny}^2 + u_1 u_{1yy} + \dots + u_n u_{nyy})$$

$$= u_{1x}^2 + \dots + u_{nx}^2 + u_{1y}^2 + \dots + u_{ny}^2 + \underbrace{u_1(\overbrace{u_{1xx} + u_{1yy}}^{0}) + \dots + u_n(\overbrace{u_{nxx} + u_{nyy}}^{0})}_{\text{since } u_i \text{ are harmonic.}}$$

$$\Rightarrow 0 = u_{1x}^2 + \dots + u_{nx}^2 + u_{1y}^2 + \dots + u_{ny}^2$$

$$\Rightarrow u_{ix} = 0 \text{ and } u_{iy} = 0 \ \forall i \ \text{(since squares are positive)}$$

$$\Rightarrow u_i \text{ const } \forall i$$

(ii) $f : D \to \mathbb{C}$ with $f(x+iy) = u(x,y) + iv(x,y)$ is <u>complex harmonic</u> if u, v are harmonic.

Show that if $f(x+iy)$ and $(x+iy)f(x+iy)$ are both c. harm. then f is analytic :

$f(x+iy) = u(x,y) + iv(x,y)$ is c-harm $\Rightarrow \begin{cases} u_{xx} + u_{yy} = 0 \\ v_{xx} + v_{yy} = 0 \end{cases}$

$$(x+iy)f(x+iy) = (x+iy)(u(x,y) + iv(x,y))$$
$$= (x+iy)u + i(x+iy)v$$
$$= xu + iyu + ixv + -yv$$
$$= (ux - vy) + i(uy + vx) \quad \text{is c. harm.}$$

$$\Rightarrow \begin{cases} (ux-vy)_{xx} + (ux-vy)_{yy} = 0 \\ (uy+vx)_{xx} + (uy+vx)_{yy} = 0 \end{cases}$$

$$\Rightarrow (ux-vy)_{xx} = (ux)_{xx} - v_{xx}y = (u_x x + u)_x - v_{xx}y$$
$$= u_{xx}x + u_x + u_x - v_{xx}y$$
$$= \underline{2u_x} + \underline{u_{xx}x} - v_{xx}y$$

$$\Rightarrow (ux-vy)_{yy} = (ux)_{yy} - (vy)_{yy} = u_{yy}x - (v_y y + v)_y$$
$$= u_{yy}x - (v_{yy}y + v_y + v_y)$$
$$= \underline{2v_y} + \underline{u_{yy}x} - v_{yy}y$$

$$\Rightarrow 2u_x + 2v_y + u_{xx}x + u_{yy}x - v_{xx}y - v_{yy}y = 0$$
$$\Rightarrow 2(u_x + v_y) + x(\underbrace{u_{xx} + u_{yy}}_{0}) - y(\underbrace{v_{xx} + v_{yy}}_{0}) = 0$$
$$\Rightarrow \underline{2(u_x + v_y) = 0}$$

<u>etc.</u> $\longrightarrow$

③ $f: D \to D$ analytic on bdd domain D with $0 \in D$ and $f(0)=0$, $|f'(0)|<1$. Let $F_n(z) = \underbrace{f \circ \cdots \circ f}_{n \text{ times}}(z)$

Show $F_n(z) \to 0$ as $n \to \infty$ unif on cpt subsets.

See that $|f'(0)|<1$

Now note that $\left| \lim_{z \to 0} \frac{f(z)-f(0)}{z-0} \right| = |f'(0)|$

hence: $\left| \frac{f(z)-f(0)}{z-0} \right| \leq |f'(0)| < 1 \Rightarrow \left| \frac{f(z)}{z} \right| \leq |f'(0)| < 1$

Now choose $|f'(0)| < \rho < 1$, and now $|f(z)| < \rho |z|$.

Now for $z_0 \in D$, let $f(z_0) = z_1$, $f(z_1) = z_2$, etc...
(i.e. $F_n(z_0) = z_n$), and we now have:

$$|z_n| = |f(z_{n-1})| \leq \rho |z_{n-1}| = \rho |f(z_{n-2})| \leq \rho^2 |z_{n-2}|$$

Hence: $\cdots \leq \rho^n |z_0|$.

$$\lim_{n \to \infty} |F_n(z_0)| = \lim_{n \to \infty} |z_n| \leq \lim_{n \to \infty} \rho^n |z_0| = 0 \text{ since } \rho < 1$$