# COMPLEX ANALYSIS GRADUATE EXAM
## Fall 2011

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

Residue ✓ Integral

(1) Evaluate

$$\int_0^{2\pi} \frac{d\theta}{3+\cos\theta+2\sin\theta}.$$

$|z|^2 = z\cdot\bar{z}$ + series mult. + Cauchy thm.

(2) Suppose the series $f(z)=\sum_{n=0}^{\infty} c_n z^n$ converges for $|z|<R$. Show that for $r<R$,

$$\int_{\{|z|=r\}} |f(z)|^2\,dz = 2\pi\sum_{n=0}^{\infty}|c_n|^2 r^{2n}.$$

Reflection ✓

(3) Let $f(z)$ be analytic on $\mathbb{C}$ and suppose that the line $\Gamma=\{t+it : t\in\mathbb{R}\}$ is mapped to itself, that is, $f(z)\in\Gamma$ for all $z\in\Gamma$. If $f(\sqrt{2})=3$, then what is $f(\sqrt{2}i)$?

Schwarz ✓ + Riemann mapping

(4) Let $\Omega\subset\mathbb{C}$, with $\Omega\neq\mathbb{C}$, be simply connected, and let $f:\Omega\to\Omega$ be a conformal bijection. If $f$ has two distinct fixed points $z_1, z_2$ (that is, $f(z_1)=z_1$, $f(z_2)=z_2$), show that $f$ is the identity map.

(1)

$\int_0^{2\pi} Q(\cos\theta,\sin\theta)\,d\theta \rightsquigarrow \cos\theta = \dfrac{z+\frac{1}{z}}{2}$, $\sin\theta = \dfrac{z-\frac{1}{z}}{2i}$

when $|z|=1$ $(z=e^{i\theta})$

$\int_{|z|=1} f(z)\,dz = \int_0^{2\pi} f(e^{i\theta})\,ie^{i\theta}\,d\theta = \int_0^{2\pi} \dfrac{Q(\cos\theta,\sin\theta)}{ie^{i\theta}}\,ie^{i\theta}\,d\theta$

$= \int_0^{2\pi} Q(\cos\theta,\sin\theta)\,d\theta$

$f(z) = \dfrac{Q\left(\frac{1}{2}(z+\frac{1}{z}), \frac{1}{2i}(z-\frac{1}{z})\right)}{iz}$



(2) $\sum_{n=0}^{\infty}\left(\sum_{i+j=n} c_i z^i\, c_j \bar{z}^j\right)$

not (square of series)

CA - Fall '11:

① $\int_0^{2\pi} \frac{d\theta}{3+\cos\theta+2\sin\theta}$

$$= \int_0^{2\pi} \frac{d\theta}{3+\frac{e^{i\theta}+e^{-i\theta}}{2}+2\left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right)} = \int_0^{2\pi} \frac{ie^{i\theta}d\theta}{ie^{i\theta}\left(3+\frac{e^{i\theta}+e^{-i\theta}}{2}+2\left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right)\right)}$$

Then let $z=e^{i\theta}$
$\Rightarrow dz = ie^{i\theta}d\theta \longrightarrow = \int_{|z|=1} \frac{dz}{iz\left(3+\frac{z+\frac{1}{z}}{2}+2\left(\frac{z-\frac{1}{z}}{2i}\right)\right)}$

$$= \int_{|z|=1} \frac{2dz}{6iz+i(z^2+1)+2(z^2-1)} = 2\int_{|z|=1} \frac{dz}{(2+i)z^2+6iz+(i-2)}$$

Now we will solve with residue integration; consider $f(z)=\frac{1}{(2+i)z^2+6iz+(i-2)}$

It will have poles at $z = \frac{-6i \pm \sqrt{-36-4(i+2)(i-2)}}{2(2+i)}$

$$= \frac{-6i\pm\sqrt{-36-4(i^2-2^2)}}{2(2+i)} = \frac{-6i\pm\sqrt{-16}}{2(2+i)} = \frac{-3i\pm 2i}{(2+i)} = \frac{-i}{2+i} \text{ or } \frac{-5i}{2+i}$$

And see that $\frac{1}{2+i} = \frac{(2-i)}{(2+i)(2-i)} = \frac{2-i}{5}$, hence the poles are:

$$-i\left(\frac{2-i}{5}\right) = \underline{\underline{\frac{-1-2i}{5}}} \text{ or } -5i\left(\frac{2-i}{5}\right) = i(i-2) = \underline{\underline{-1-2i}}$$

Clearly only $\frac{-1-2i}{5} \in \{|z|<1\}$, hence (letting $a=\frac{-1-2i}{5}$; $b=-1-2i$)

$$\underline{\underline{\int_{|z|=1} f(z)dz = 2\pi i\left(\operatorname{Res}_{z=-\frac{(1+2i)}{5}} f(z)\right)}}$$

Now see that: $\operatorname{Res}_{z=a} f(z) = (z-a)f(z)\Big|_{z=a} = \frac{\cancel{(z-a)}}{(2+i)\cancel{(z-a)}(z-b)}\Big|_{z=a}$

$$= \frac{1}{(2+i)(a-b)} = \frac{1}{(2+i)\left(\frac{-1-2i}{5}+(1+2i)\right)} = \left(\frac{1}{2+i}\right)\left(\frac{1}{\frac{4}{5}+\frac{8}{5}i}\right)$$

$$= \left(\frac{1}{2+i}\right)\left(\frac{5}{4+8i}\right) = \left(\frac{2-i}{5}\right)\left(\frac{5}{4+8i}\right) = \frac{2-i}{4+8i} = \frac{(2-i)(4-8i)}{16+64}$$

$$= \frac{8-16i-4i-8}{80} = \frac{-20i}{80} = \frac{-i}{4}$$

So now $\int_0^{2\pi} \frac{d\theta}{3+\cos\theta+2\sin\theta} = 2\int_{|z|=1} f(z) = 2\left(2\pi i\left(\frac{-i}{4}\right)\right) = \frac{-4\pi i^2}{4} = \pi$

② $f(z) = \sum_{n=0}^{\infty} c_n z^n$ converges for $|z| < R$. Show, for $r < R$,

$$\int_{|z|=r} |f(z)|^2 dz = 2\pi \sum_{n=0}^{\infty} |c_n|^2 r^{2n} :$$

Recall that $|f(z)|^2 = f(z) \cdot \overline{f(z)}$ ; now we have:

$$\int_{|z|=r} |f(z)|^2 dz = \int_{|z|=r} \left(\sum_{n=0}^{\infty} c_n z^n\right)\left(\sum_{n=0}^{\infty} \overline{c_n}\,\overline{z}^n\right) dz$$

$$= \int_{|z|=r} \sum_{n=0}^{\infty} \left(\sum_{i+j=n}^{n} c_i z^i \overline{c_j}\,\overline{z}^j\right) dz$$

since inside radius of convergence

$$= \sum_{n=0}^{\infty} \int_{|z|=r} \left(\sum_{i+j=n} c_i \overline{c_j}\, z^i \overline{z}^j\right) dz$$

$$= \sum_{n=0}^{\infty} \int_{|z|=r} \left(\sum_{i<j} c_i \overline{c_j}\, z^i \overline{z}^j + \sum_{i>j}^{r} c_i \overline{c_j}\, z^i \overline{z}^j + c_n \overline{c_n}\, z^n \overline{z}^n\right) dz$$

$$= \sum_{n=0}^{\infty} \left(\sum_{i<j} c_i \overline{c_j} |z|^i \int_{|z|=r} \overline{z}^{j-i} dz + \sum_{i>j} c_i \overline{c_j} |z|^j \int_{|z|=r} z^{i-j} dz + c_n \overline{c_n} \int_{|z|=r} |z|^{2n} dz\right)$$

$$= \sum_{n=0}^{\infty} \left(|c_n|^2 r^{2n} 2\pi + \underbrace{\sum_{i<j} c_i \overline{c_j}\, r^i \int_{|z|=r} \overline{z}^{j-i} dz}_{0} + \cancel{\sum_{i>j} c_i \overline{c_j}\, r^j \int_{|z|=r} z^{i-j} dz}^{\,0}\right)$$

$i<j \Rightarrow 0<j-i$
$\Rightarrow$ integral is zero by Cauchy thm.

$i>j \Rightarrow 0<i-j$
$\Rightarrow$ integral is zero by Cauchy thm

$$= \sum_{n=0}^{\infty} \left(2\pi |c_n|^2 r^{2n}\right) = \boxed{2\pi \sum_{n=0}^{\infty} |c_n|^2 r^{2n}}$$

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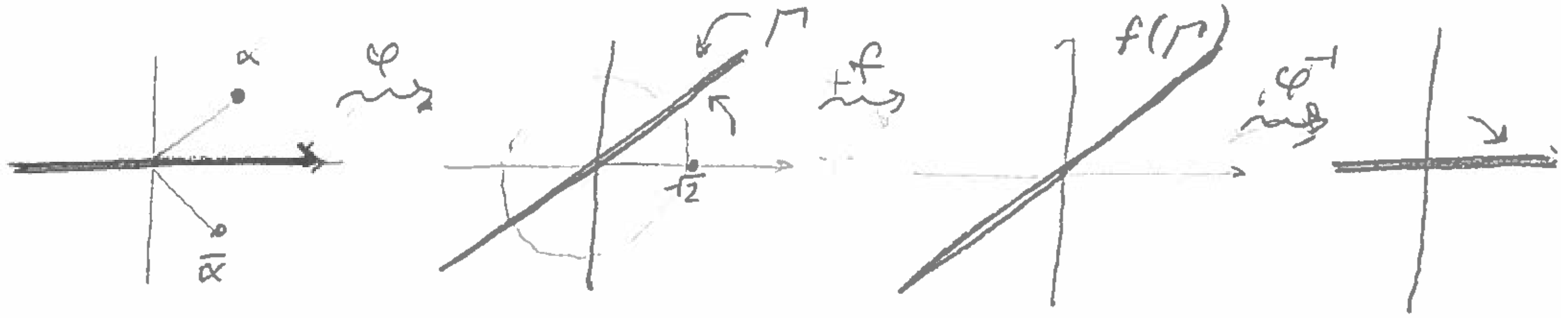
Recall:

$$\boxed{\text{Series mult: } \left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) = \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} a_i b_{n-i}\right)}$$

(3.) f analytic on $\mathbb{C}$, maps $\Gamma = \{t+it : t\in\mathbb{R}\}$ to itself. If $f(\sqrt{2})=3$, then what is $f(\sqrt{2}i)$?

Consider the rotation $\varphi(z) = ze^{i\frac{\pi}{4}}$,




So now we can see that $\varphi^{-1}\circ f\circ\varphi$ fixes the real axis (and is clearly analytic on the UHP), hence we may apply the reflection principle, namely:

$$\varphi^{-1}\circ f\circ\varphi(\bar z) = \overline{\varphi^{-1}\circ f\circ \varphi(z)}$$

Now let $\alpha\in\mathbb{C}$ be such that $\varphi(\bar\alpha)=\sqrt2$, hence $\varphi(\alpha)=i\sqrt2$

Now see that

$$\varphi^{-1}\circ f\circ\varphi(\bar\alpha) = \varphi^{-1}\circ f(\sqrt2) = \varphi^{-1}(3), \text{ and}$$

$$\varphi^{-1}\circ f\circ\varphi(\bar\alpha) = \overline{\varphi^{-1}\circ f\circ\varphi(\alpha)} = \overline{\varphi^{-1}\circ f(i\sqrt2)}$$

See that $\varphi^{-1}(z) = e^{-i\frac{\pi}{4}}z$, hence

$$3e^{-i\frac{\pi}{4}} = \overline{f(i\sqrt2)e^{-i\frac{\pi}{4}}} = \overline{f(i\sqrt2)}\,e^{i\frac{\pi}{4}}$$

$$\Rightarrow \overline{f(i\sqrt2)} = 3e^{-i\frac{\pi}{2}} = -3i$$

$$\Rightarrow \boxed{f(i\sqrt2)=3i}$$

(4.) $\Omega \subsetneq \mathbb{C}$, $\Omega \neq \mathbb{C}$, $f: \Omega \to \Omega$ conformal bijection.
If $f$ has 2 fixed pts $z_1, z_2$, then show that $f$ is the ident.

By Riemann Mapping Theorem, $\exists$ bijective, biholomorphic map $\varphi: \Omega \to D$ such that $\varphi(z_1) = 0$.

Then $\varphi \circ f \circ \varphi^{-1}(0) = \varphi \circ f(z_1) = \varphi(z_1) = 0$, and since $\varphi \circ f \circ \varphi^{-1}: D \to D$, we may apply Schwarz Lemma; hence we get:

$$|\varphi \circ f \circ \varphi^{-1}(z)| \leq |z|.$$

Now consider the point $\varphi(z_2)$; then

$$|\varphi \circ f \circ \varphi^{-1}(\varphi(z_2))| = |\varphi \circ f(z_2)| = |\varphi(z_2)|,$$

hence, again by Schwarz lemma, $\varphi \circ f \circ \varphi^{-1}(z) = cz$, $|c| = 1$.

Now we also have:

$$\varphi \circ f \circ \varphi^{-1}(z_2) = cz_2 \Rightarrow \varphi \circ f(z_2) = c\varphi(z_2)$$

$$\Rightarrow \varphi^{-1}(c\varphi(z_2)) = f(z_2) = z_2$$

$$\Rightarrow \varphi(z_2) = c\varphi(z_2) \Rightarrow c = 1 \text{ since } \varphi(z_2) \neq 0$$

Therefore $\varphi \circ f \circ \varphi^{-1}(z) = z$

$$\Rightarrow \varphi \circ f(z) = \varphi(z)$$

$$\Rightarrow \underline{\underline{f(z) = z}}, \text{ i.e. } f \text{ identity}$$