COMPLEX ANALYSIS GRADUATE EXAM
Spring 2010

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

Notation: $\Re z$ denotes the real part of the complex number z, and $\Im z$ its imaginary part.

(1) Map the region $\Omega = \{\Im z > 0\} \setminus \{iy : 0 < y \leq 1\}$ conformally to the unit disk $D = \{|z| < 1\}$.

(2) How many zeroes of $p(z) = z^4 + z^3 - 4z^2 + 2z + 7$ lie in the right half plane $\{\Re z > 0\}$?

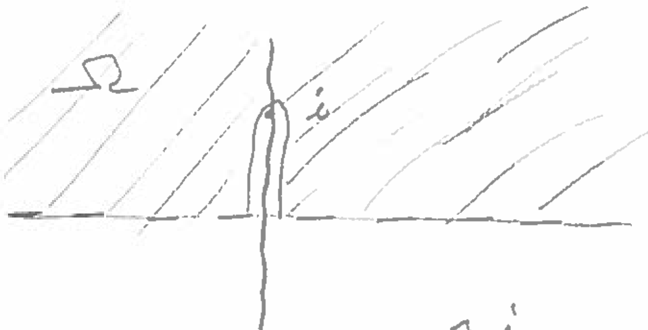
(3) Let f be analytic in the unit disk $D = \{|z| < 1\}$ and continuous on its closure $\bar{D}$. Show that if f is real valued on the boundary $\partial D = \{|z| = 1\}$ then f must be a constant.

(4) By consideration of $\int e^{z+\frac{1}{z}}\,dz$, or otherwise, show that

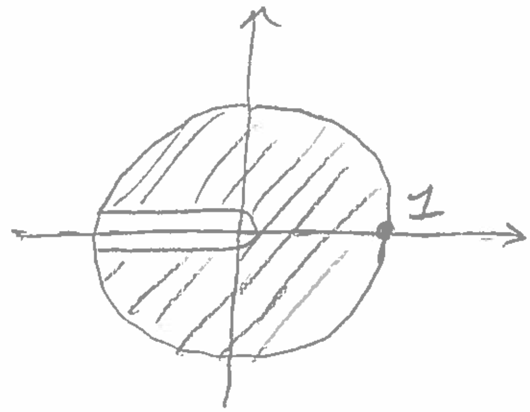
$$\frac{1}{2\pi}\int_0^{2\pi} e^{2\cos\theta}\cos\theta\,d\theta = 1 + \frac{1}{2!} + \frac{1}{2!3!} + \frac{1}{3!4!} + \cdots.$$

CA-Spring '10

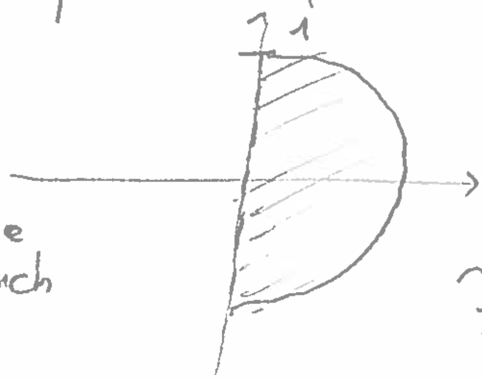
(1.) Conformally map $\Omega = \{\mathrm{Im}(z) > 0\} \setminus \{iy : 0 < y \le 1\}$ to the unit disc



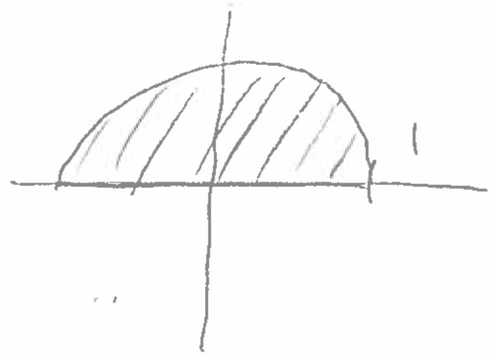
$\xrightarrow{\frac{z-i}{z+i}}$



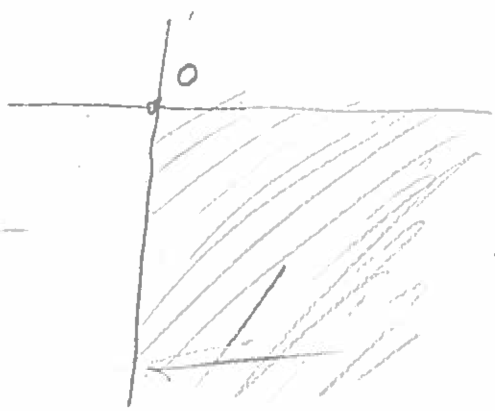
$\xrightarrow{z^{1/2}}$ (defined since we have branch cut)



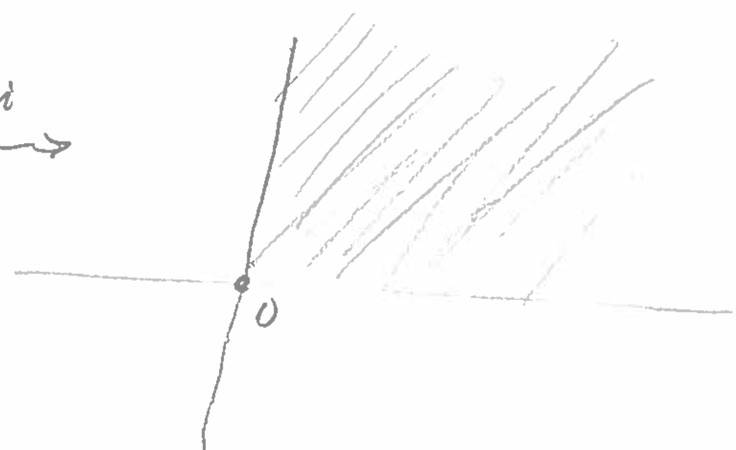
$\xrightarrow{zi}$



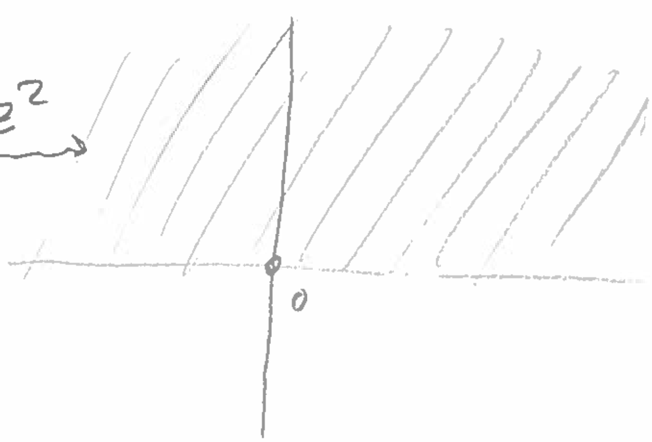
$\xrightarrow{\frac{1-z}{1+z}}$



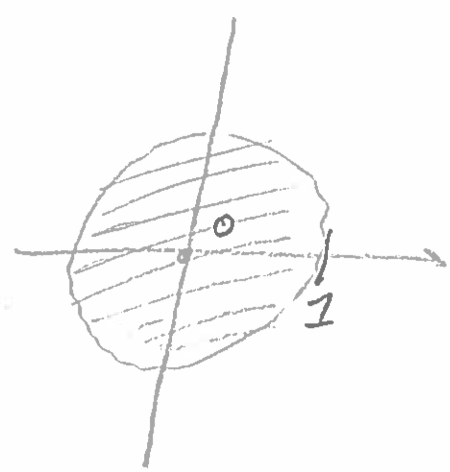
$\xrightarrow{zi}$



$\xrightarrow{z^2}$

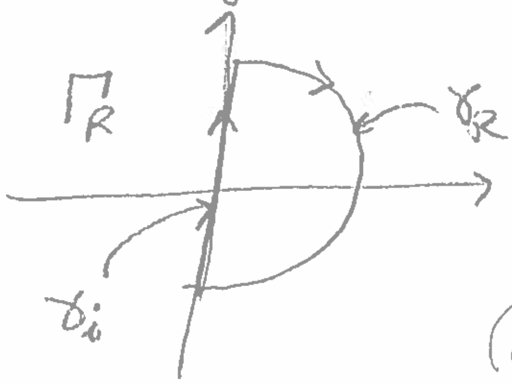


$\xrightarrow{\frac{1-z}{1+z}}$



(2.) How many zeroes of $p(z) = z^4 + z^3 + 4z^2 + 2z + 7$ in the RHP?

Consider the contour Γ_R with R large enough to enclose all zeroes of p in the RHP:



Now recall that

$\#$ zeroes of p inside $\Gamma_R = n(p(\Gamma_R), 0)$,

hence $= \frac{1}{2\pi}\left(\begin{array}{c}\text{change in arg } p(z) \\ \text{as } z \text{ traverses } \Gamma_R\end{array}\right)$

(Every change 2π = one wind about 0)

- Consider γ_i: here $z = iy$ for $y \in \mathbb{R}$, hence

$$p(iy) = y^4 - iy^3 - 4y^2 + 2iy + 7 = (y^4 - 4y^2 + 7) + (2y - y^3)i$$

now if $y^4 - 4y^2 + 7 = 0$, then $y^2 = \frac{4 \pm \sqrt{16 - 4(1)(7)}}{2(1)}$, hence only complex roots, hence $y^4 - 4y^2 + 7 \neq 0$ for all $y \in \mathbb{R}$ hence since $0^4 - 4(0)^2 + 7 = 7 > 0$, we have $p(iy)$ have positive real part for all y, hence $p(\gamma_i)$ has zero winding number about 0.

- Consider γ_R: Consider $z \in \gamma_R$; then $z = Re^{i\theta}$ with $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ hence as z traverses γ_R there is a change in $\arg z$ of π.

Now see that:

$$p(Re^{i\theta}) = R^4 e^{4i\theta} + R^3 e^{3i\theta} + 4R^2 e^{2i\theta} + 2e^{i\theta} + 7$$

$$= R^4 e^{4i\theta}\left(1 + \frac{1}{Re^{i\theta}} + \frac{4}{R^2 e^{2i\theta}} + \frac{2}{R^3 e^{3i\theta}} + \frac{7}{R^4 e^{4i\theta}}\right)$$

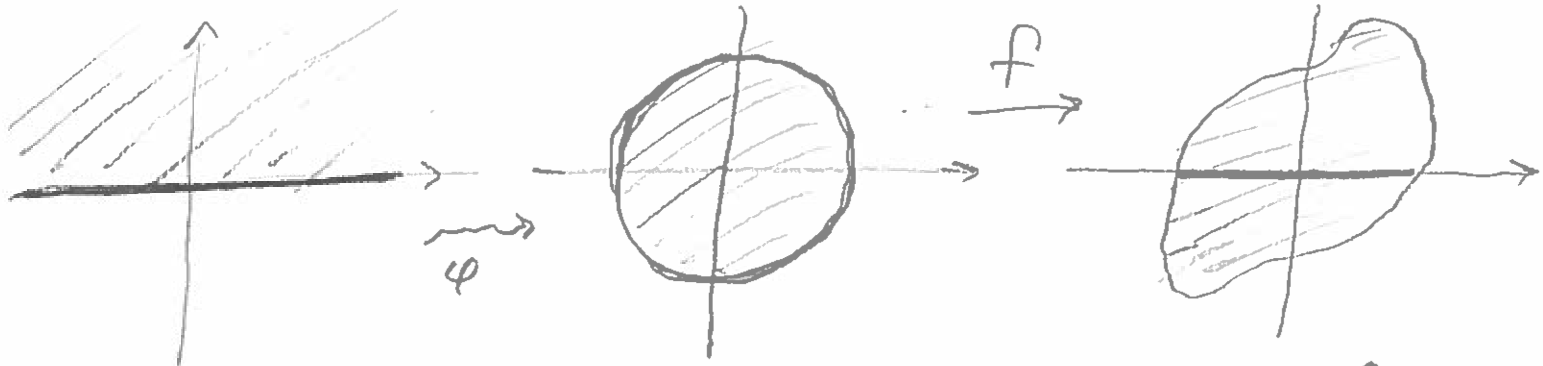
, i.e. $R^4 e^{4i\theta}$ dominates for R large, hence for R large & $z \in \gamma_R$ $\arg p(z) = 4\theta$, hence the change in $\arg p(z)$ is approx 4π.

$\Rightarrow$ # zeroes in RHP $= \frac{1}{2\pi}(4\pi) = 2$

(3.) f analytic in $D = \{|z|<1\}$ and cont. on $\overline{D}$.

Show that if f is $\mathbb{R}$-valued on ∂D then $f \equiv$ const.

Consider the map $\varphi(z) = \frac{i-z}{i+z}$ which sends $UHP \to D$, $\mathbb{R} \to \partial D$



So we have the fn. $f\circ\varphi(z)$ sending the real line $\to$ real line and analytic on the UHP, hence apply the Reflection Principle:

$f\circ\varphi$ extends to analytic function on entire plane

and $f\circ\varphi(\bar{z}) = \overline{f\circ\varphi(z)}$

Now, φ is bounded by construction and f is bounded on $\overline{D}$ since f is continuous and $\overline{D}$ is compact.

Therefore $f\circ\varphi$ is bounded

But we just showed $f\circ\varphi$ is entire, hence by Liouville

$f\circ\varphi$ is constant.

Therefore f is constant since φ was not.

(4.) Show that $\frac{1}{2\pi}\int_0^{2\pi} e^{2\cos\theta}\cos\theta\, d\theta = 1 + \frac{1}{2!} + \frac{1}{2!3!} + \frac{1}{3!4!} + \cdots$

(consider $\int e^{z+\frac{1}{z}} dz$)

Consider the integral $\int_{|z|=1} e^{z+\frac{1}{z}} dz = \int_0^{2\pi} e^{e^{i\theta}+e^{-i\theta}} ie^{i\theta} d\theta$

$= i\int_0^{2\pi} e^{2\cos\theta}(\cos\theta + i\sin\theta)\, d\theta = i\int_0^{2\pi} e^{2\cos\theta}\cos\theta\, d\theta + i\int_0^{2\pi} e^{2\cos\theta}\sin\theta\, d\theta$

So now apply the residue theorem to $\int e^{z+\frac{1}{z}} dz$; $e^{z+\frac{1}{z}}$ has one pole at $z=0$, so then compute:

$\underset{z=0}{\mathrm{Res}}\, e^{z+\frac{1}{z}}$: $e^{z+\frac{1}{z}} = e^z e^{\frac{1}{z}} = \left(\sum_{n=0}^{\infty} \frac{z^n}{n!}\right)\left(\sum_{m=0}^{\infty} \frac{1}{m!}\left(\frac{1}{z^m}\right)\right)$

$= \sum_{n=0}^{\infty}\left(\sum_{i+j=n} \frac{z^i}{i!}\cdot\frac{1}{j!}\frac{1}{z^j}\right)$

Now, we are interested in the coefficients of $\frac{z^n}{z^m}$ when $m=n+1$, ranging over all m, n,

i.e. $\left(\frac{z^n}{n!}\right)\left(\frac{1}{m!\,z^m}\right) = \frac{1}{n!(n+1)!\,z}$; n ranges from n to ∞,

So our residue is $\sum_{i=0}^{\infty} \frac{1}{n!(n+1)!} = 1 + \frac{1}{2!} + \frac{1}{2!3!} + \cdots$

Hence $\int_{|z|=1} e^{z+\frac{1}{z}} dz = 2\pi i\left(\underset{z=0}{\mathrm{Res}}\, e^{z+\frac{1}{z}}\right] = 2\pi i\left(\sum_{i=0}^{\infty} \frac{1}{n!(n+1)!}\right)$

$\Rightarrow 2\pi i\left(\sum_{i=0}^{\infty} \frac{1}{n!(n+1)!}\right) = i\int_0^{2\pi} e^{2\cos\theta}\cos\theta\, d\theta - \int_0^{2\pi} e^{2\cos\theta}\sin\theta\, d\theta$

Then take the imaginary pt. of both sides to get

$$2\pi\left(\sum_{i=0}^{\infty} \frac{1}{n!(n+1)!}\right) = \int_0^{2\pi} e^{2\cos\theta}\cos\theta\, d\theta$$