COMPLEX ANALYSIS GRADUATE EXAM

Fall 2010

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

+✓ 1. Show that

Half-residue

$$\int_0^\infty \frac{\sin x}{x(x^2+1)}\,dx = \frac{\pi(1-e^{-1})}{2}.$$

+✓ 2. Suppose that f is holomorphic in a neighborhood of 0 and that

Taylor converges for all z

$$(*) \qquad \sum_{n=0}^{\infty} f^{(n)}(z)$$

is absolutely convergent at $z = 0$. Show that f is an entire function, and that $(*)$ is convergent for all $z \in \mathbb{C}$.

Poisson formula — 3. Let f be a non-negative real valued harmonic function in the disc $D = \{z \in \mathbb{C} : |z| < R\}$.

(i) Prove that

$$\frac{R-|z|}{R+|z|}f(0) \le f(z) \le \frac{R+|z|}{R-|z|}f(0) \quad \text{whenever } |z| < R.$$

[Hint: use the Poisson formula.]

(ii) Prove that

$$\frac{1}{3}f(0) \le f(z) \le 3f(0) \quad \text{whenever } |z| \le R/2.$$

(iii) Let K be a compact subset of the open disc D. Show that there is a constant M depending only on K and R such that

$$f(z_1) \le M f(z_2) \quad \text{for all } z_1, z_2 \in K.$$

Cauchy Theorem ✓ ✓ 4. Liouville's theorem states that a bounded entire function f is constant.

i) Give a proof of Liouville's theorem. You may use standard results about holomorphic functions such as Cauchy's theorem and power series representation, but any result you use should be clearly stated

(ii) Suppose instead that f is entire and that $|f(z)| \le K(1+|z|^n)$ for some $K < \infty$ and positive integer n. Show that f is a polynomial of degree at most n.

② $|f(z)| = \left|\sum_{n=0}^{\infty} \frac{f^{n}(0)}{n!} z^n\right| \leq \sum_{n=0}^{\infty} |f^{(n)}(0)| \left|\frac{z^n}{n!}\right|$

But $\lim_{n\to\infty} \frac{|z|^n}{n!} = 0$, hence $\left|\frac{z^n}{n!}\right| \leq C(z)$

bd depends only on z, not n.

$\leq \sum_{n=0}^{\infty} C(z) |f^{n}(0)|$

$= C(z) \sum_{n=0}^{\infty} |f^{n}(0)|$

finite

finite by hypothesis.

So taylor series converges for all $z \Rightarrow$ entire

$\left|\frac{z^n}{n!}\right| \leq \sum_{n=0}^{\infty} \frac{|z|^n}{n!} = e^{|z|}$

So get bd. only dependent on z.

CA - Fall '10

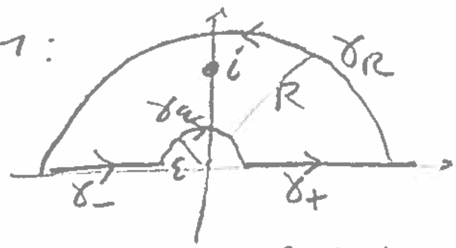
(1.) Show $\int_0^\infty \frac{\sin x}{x(x^2+1)}dx = \frac{\pi(1-e^{-1})}{2}$.

Since $\frac{\sin(-x)}{-x((-x)^2+1)} = \frac{-\sin(x)}{-x(x^2+1)} = \frac{\sin x}{x(x^2+1)}$, the integrand is even

and $\int_0^\infty \frac{\sin x}{x(x^2+1)}dx = \frac{1}{2}\int_{-\infty}^\infty \frac{\sin x}{x(x^2+1)}$

Now consider the function $f(z) = \frac{e^{iz}}{z(z^2+1)}$ integrated over the contour

Γ:



see that $\int_{\gamma_1} + \int_{\gamma_+} \longrightarrow \int_{-\infty}^\infty \frac{e^{iz}}{z(z^2+1)}dz$ for $R \to \infty$ and $\varepsilon \to 0$.

- Consider first $\int_{\gamma_R} f(z)dz$:

$\theta \in (0,\pi)$ hence $\sin\theta > 0$ for all θ

$$\left|\int_{\gamma_R} f(z)dz\right| \leq \int_{\gamma_R} \frac{|e^{iz}|}{|z||z^2+1|}|dz| \leq \int \frac{e^{-R\sin\theta}}{R(R^2-1)}|dz| \leq \frac{\pi R}{R(R^2-1)} \to 0$$
as $R \to \infty$

- Now consider $\int_{\gamma_\varepsilon} f(z)dz$:

Since f has a simple pole at $z=0$, we may rewrite $f(z) = \frac{a}{z} + h(z)$ for h holomorphic at 0 (note that $a = \operatorname{Res}_{z=0} f(z)$)

Then we have:
$$\int_{\gamma_\varepsilon} f(z)dz = \int_{\gamma_\varepsilon} \frac{a}{z}dz + \int_{\gamma_\varepsilon} h(z)dz.$$

Now, h is holomorphic at 0, hence bdd there, and as $\varepsilon \to 0$, length$(\gamma_\varepsilon) \to 0$, hence $\int_{\gamma_\varepsilon} h(z)dz \to 0$ as $\varepsilon \to 0$.

So as $\varepsilon \to 0$, $\int_{\gamma_\varepsilon} f(z)dz \longrightarrow \int_{\gamma_\varepsilon} \frac{a}{z}dz = \underline{-\pi i a}$ (oriented CW)

- So now we have, as $R \to \infty$, $\varepsilon \to 0$,

$$\int_\Gamma f(z)dz = \int_{-\infty}^\infty \frac{e^{iz}}{z(z^2+1)}dz - \pi i a \Rightarrow \int_{-\infty}^\infty \frac{e^{iz}}{z(z^2+1)}dz = \int_\Gamma f(z)dz + \pi i a$$

So now we'll compute $\int_{\Gamma} f(z)\,dz$ with the residue theorem:

$$\int_{\Gamma} f(z)dz = 2\pi i\left(\operatorname*{Res}_{z=i} f(z)\right)$$

and $\operatorname*{Res}_{z=i} f(z) = \left.\dfrac{e^{iz}(z-i)}{z(z^2+1)}\right|_{z=i} = \dfrac{e^{i(i)}}{i(i+i)} = \dfrac{e^{-1}}{-2}$

hence $\int_{\Gamma} f(z)dz = -2\pi i\left(\dfrac{e^{-1}}{2}\right)$

hence $\displaystyle\int_{-\infty}^{\infty} \frac{e^{iz}}{z(z^2+1)}dz = -2\pi i\left(\frac{e^{-1}}{2}\right) + \pi i \operatorname*{Res}_{z=0} f(z)$

so see then $\operatorname*{Res}_{z=0} f(z) = \dfrac{e^{i(0)}}{(0^2+1)} = \dfrac{1}{1} = 1$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\cos z + i\sin z}{(z)(z^2+1)} = -\cancel{2}\pi i\left(\frac{e^{-1}}{\cancel{2}}\right) + \pi i$$

$$= -\pi i e^{-1} + \pi i = \pi i(1-e^{-1})$$

Taking imaginary part:

$$\int_{-\infty}^{\infty} \frac{\sin z\, dz}{z(z^2+1)} = -\pi(1-e^{-1})$$

$$\Rightarrow \boxed{\int_{0}^{\infty} \frac{\sin z}{z(z^2+1)}dz = \frac{\pi(1-e^{-1})}{2}}$$

2. f holomorphic in nbhd of 0 and $\sum_{n=0}^{\infty} f^{(n)}(0)$ absolutely conv.
Show f is entire and $\sum_{n=0}^{\infty} f^{(n)}(z)$ convergent for all z.

If the Taylor series for f converges for all z, then f is entire.
So now consider the following:

$$|f(z)| = \left| \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n \right| \leq \sum_{n=0}^{\infty} |f^{(n)}(0)| \left| \frac{z^n}{n!} \right|$$

Now note that $\left| \frac{z^n}{n!} \right| \leq \sum_{i=0}^{\infty} \frac{|z|^n}{n!} = e^{|z|}$,
hence $\left| \frac{z^n}{n!} \right| \leq e^{|z|}$ for all z, i.e. it is bounded by a constant dependent only on z.

$$\Rightarrow \sum_{n=0}^{\infty} |f^{(n)}(0)| \left| \frac{z^n}{n!} \right| \leq \underbrace{e^{|z|}}_{\text{finite for all } z} \overbrace{\sum_{n=0}^{\infty} |f^{(n)}(0)|}^{\text{finite by hypothesis}} < \infty$$

Hence $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$ convergent for all z, hence f entire.

~~Since f is entire, we can write its Taylor series about any $z_0 \in \mathbb{C}$, i.e.~~

$$\cancel{f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z-z_0)^n \quad \text{is convergent}}$$

~~Now consider~~ $\cancel{f(z_0+1) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}}$

(3.) f non-negative, $\mathbb{R}$-valued, harmonic in $D_R = \{|z|<R\}$.

(i) Prove: $\frac{R-|z|}{R+|z|} f(0) \le f(z) \le \frac{R+|z|}{R-|z|} f(0)$ whenever $|z|<R$

Recall the Poisson formula: $\boxed{f(z) = \int_{|w|=R} f(w)\, Re\left(\frac{w+z}{w-z}\right) \frac{d\theta}{2\pi}}$

So now we have:

$$|f(z)| \le \int_{|w|=R} |f(w)| \left| Re\left(\frac{w+z}{w-z}\right)\right| \frac{d\theta}{2\pi} \qquad (w = Re^{i\theta})$$

$$\le \int_{|w|=R} |f(w)| \left|\frac{w+z}{w-z}\right| \frac{d\theta}{2\pi} \le \int_{|w|=R} |f(w)| \left(\frac{|w|+|z|}{|w|-|z|}\right) \frac{d\theta}{2\pi}$$

$$= \int_{|w|=R} |f(w)| \left(\frac{R+|z|}{R-|z|}\right) \frac{d\theta}{2\pi} = \left(\frac{R+|z|}{R-|z|}\right) \int_{|w|=R} |f(w)| \frac{d\theta}{2\pi}$$

f non-neg

$$= \frac{R+|z|}{R-|z|} \int_{|w|=R} f(w) \frac{d\theta}{2\pi} = \frac{R+|z|}{R-|z|} f(0)$$

by Poisson formula

$$\Rightarrow \quad -\left(\frac{R+|z|}{R-|z|}\right) f(0) \le f(z) \le \frac{R+|z|}{R-|z|} f(0)$$

$$\frac{1}{2\pi} Re\left(\frac{Re^{i\theta}+r}{Re^{i\theta}-r}\right)$$

?

(4) (i) Prove Liouville's theorem
(ii) Prove generalized Liouville, i.e. $|f(z)| \le k(1+|z|^n) \Rightarrow f$ poly of degree $\le n$.

(i) Let f be entire and $|f(z)| \le M$ for all z; now consider Cauchy:

$$|f^{(n)}(0)| = \left|\frac{1}{2\pi i}\int_{|\xi|=r} \frac{f(\xi)}{(\xi)^{n+1}}\,d\xi\right| \le \frac{1}{2\pi}\int_{|\xi|=r} \frac{|f(\xi)|}{|\xi|^{n+1}}\,|d\xi|$$

$$\le \frac{1}{2\pi}\int_{|\xi|=r} \frac{M}{r^{n+1}}\,|d\xi| = \frac{M}{2\pi r^{n+1}}(2\pi r) = \frac{M}{r^n}$$

But since f is entire, this is true for all r, hence letting $r \to \infty$, we see that $|f^{(n)}(0)| = 0$ for $n \ge 1$.

Hence in the Taylor series for f;

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$$

All terms are 0 except at $n=0$, hence $f(z) = f(0)$, a constant.

(ii) Let f be entire and $|f(z)| \le K(1+|z|^n)$

Now consider Cauchy:

$$|f^{(n+1)}(0)| = \left|\frac{1}{2\pi i}\int_{|\xi|=r} \frac{f(\xi)}{(\xi)^{n+2}}\,d\xi\right| \le \frac{1}{2\pi}\int_{|\xi|=r} \frac{|f(\xi)|}{|\xi|^{n+2}}\,|d\xi|$$

$$\le \frac{K(1+|z|^n)}{2\pi r^{n+2}}\int_{|\xi|=r} |d\xi| = \frac{K(1+|z|^n)}{2\pi r^{n+2}}\,2\pi r = \frac{K(1+|z|^n)}{r^{n+1}}$$

Again, since f is entire this is valid for any r, hence let $r \to \infty$ and we get $|f^{(n+1)}(0)| = 0$

Therefore the Taylor series has 0 terms for all terms $n+1$ and greater, i.e. f is at most a degree n polynomial.