

shitty exam

COMPLEX ANALYSIS GRADUATE EXAM
Spring 2009

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

✓ (1) Map the region $\Omega = \{|z| < 1\} \setminus [\frac{1}{2}, 1)$ conformally to the unit disk $D = \{|z| < 1\}$.

~~(2)~~ Find a bounded harmonic function $\phi(x, y)$ in the region $\Omega = \{(x, y) : x^2 > y^2 - 1\}$ that satisfies the boundary condition $\phi(x, y) = 1$ if $x^2 = y^2 + 1, y > 0$ and $\phi(x, y) = -1$ if $x^2 = y^2 + 1, y < 0$.

TYP0

(3) Evaluate the integral

$$\int_0^{\infty} \frac{\log^2 x}{1+x^2} dx.$$

May work for
 $x > y^2 + 1$

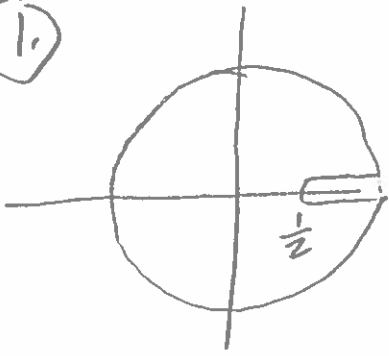
~~(4)~~ Find the number of roots of the equation

$$z^6 - 5z^4 + 8z - 1 = 0$$

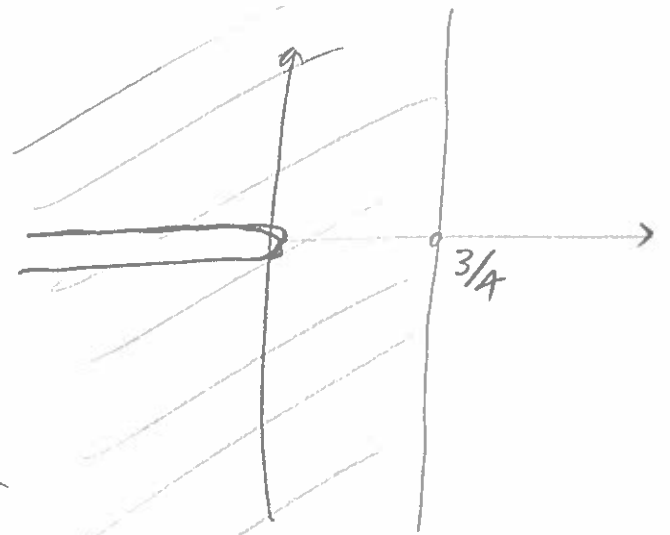
TYP0

in the annulus $\{1 < |z| < 2\}$.

1.

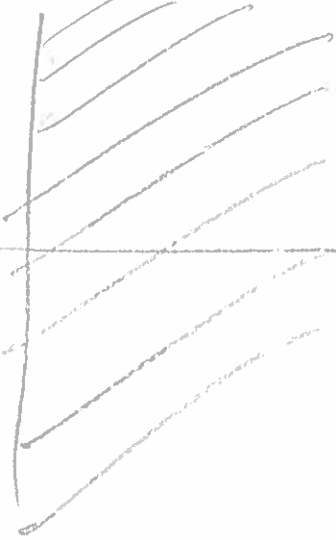


$$\frac{z - \frac{1}{2}}{z - 1}$$

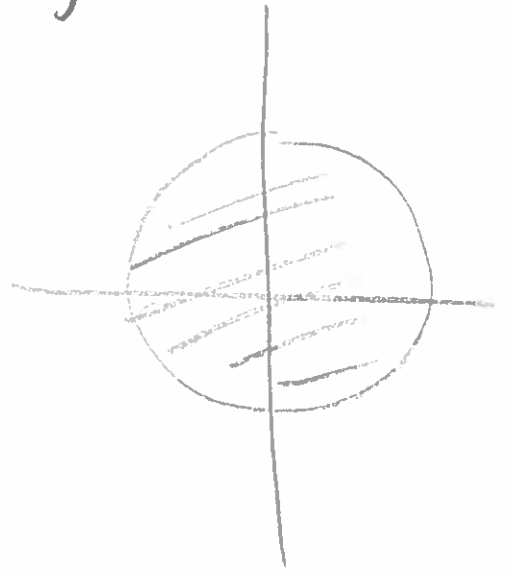


\sqrt{z}

ok because of branch cut on neg. real axis



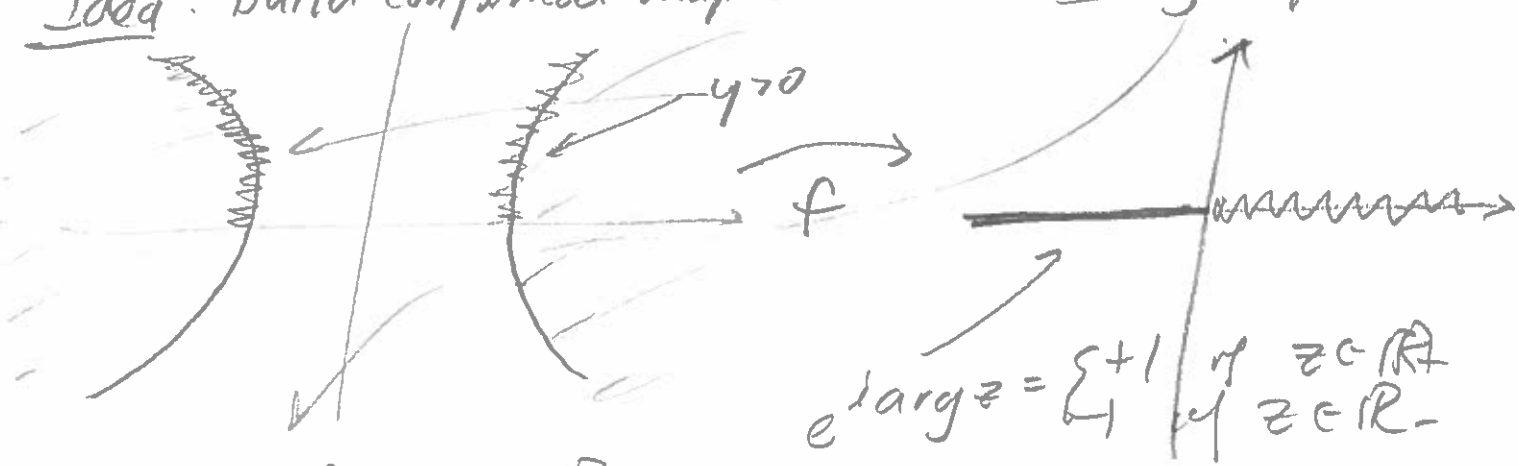
$$\frac{z-1}{z+1}$$



② Find hdd harm $\phi(x,y)$ in $\Omega = \{(x,y) = x^2 > y^2 + 1\}$

s.t. $\phi(x,y) = 1$ on $\{x^2 = y^2 + 1, y > 0\}$
 $\phi(x,y) = -1$ on $\{x^2 = y^2 + 1, y < 0\}$

Idea: Build conformal map f with the following properties:



$$e^{i \arg z} = \begin{cases} +1 & \text{if } z \in \mathbb{R}^+ \\ -1 & \text{if } z \in \mathbb{R}^- \end{cases}$$

$$\begin{cases} x^2 = y^2 + 1 \rightsquigarrow \mathbb{R} \\ y > 0 \rightsquigarrow \mathbb{R}^+ \\ y < 0 \rightsquigarrow \mathbb{R}^- \end{cases}$$

$$e^{i \arg f(z)} = \begin{cases} +1 & y > 0 \\ -1 & y < 0 \end{cases}$$

on $x^2 = y^2 + 1$.

so let $\phi = \frac{e^{i \arg f}}{e}$

$$z = x + iy \in \{x^2 = y^2 + 1\} = \{x^2 - y^2 = 1\}$$

$$\begin{aligned} z^2 &= (x + iy)^2 = x^2 - y^2 + 2xyi \\ &= 1 + 2xyi \Rightarrow \operatorname{Re}(z^2) = 1. \end{aligned}$$

