COMPLEX ANALYSIS GRADUATE EXAM
Fall 2009

Answer all five questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

Conformal ✓ (1) Map the region $\Omega = \{z = x + iy : |z| < 1, y > 1/\sqrt{2}\}$ conformally to the unit disk $D = \{z : |z| < 1\}$.

Branch cuts ✓ (2) Let $f(z)$ be an entire function so that $f(z)$ does not assume any value in $[0, \infty)$. Show that f is a constant.

Residue — (3) Evaluate the integral $\displaystyle\int_0^\infty \frac{dx}{(1-x^2)\sqrt{x}}$.

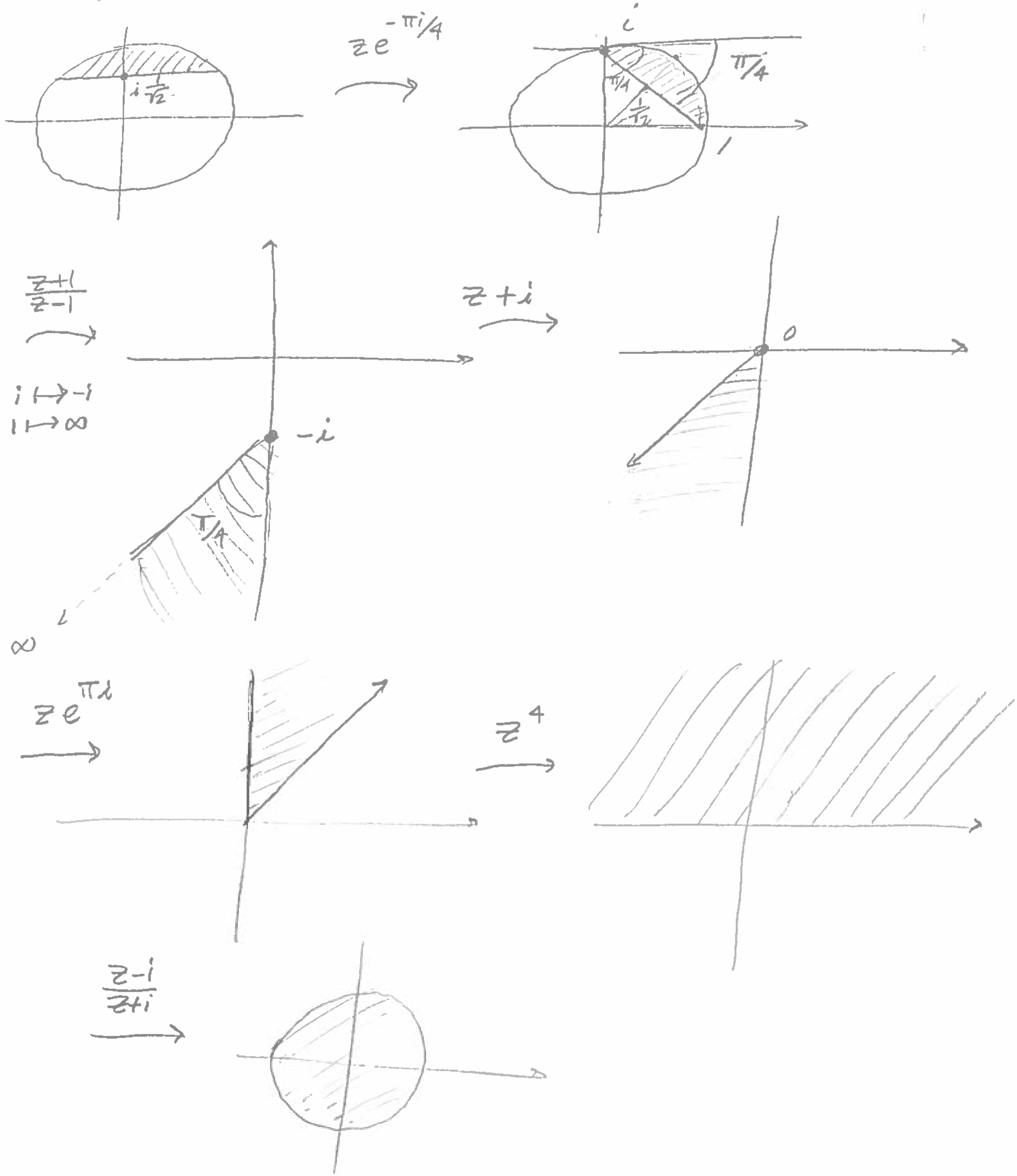
Expand, etc. — ✓ (4) Show that the infinite product $\displaystyle\prod_{k=0}^{\infty}(1 + z^{2^k})$ converges for $|z| < 1$ and equals $\dfrac{1}{1-z}$ (for $|z| < 1$).

Schwartz Lemma ✓ (5) Show that if $f : D \to D$ (D the unit disk) has two distinct fixed points, then $f(z) = z$.

$|a| = |a+b-b| \le |a+b| + |b|$

CA - Fall 09

(1.) Conformally map $\Omega = \{|z|<1,\ y > \frac{1}{\sqrt{2}}\}$ to $D = \{|z|<1\}$



② f entire, $f(z) \notin [0,\infty)$. Show that f is constant.

We see first that the image of f is contained in the region:

Hence the logarithm is well-defined in $im(f)$, hence the square root is well defined:

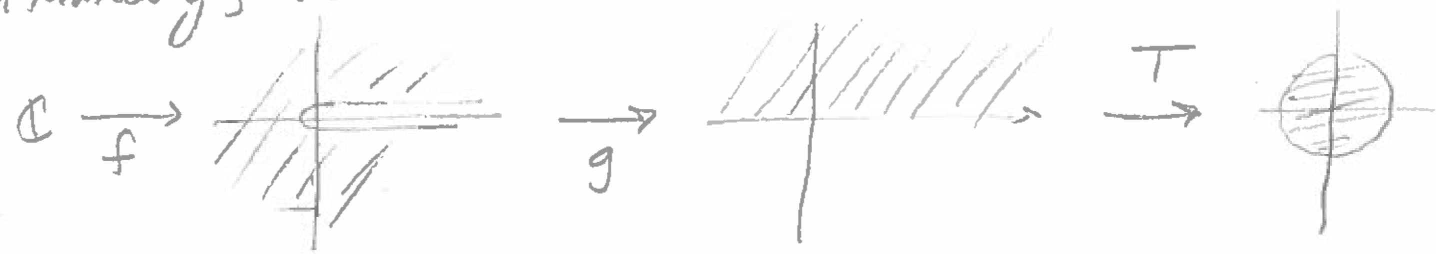
Let $g(z) = \sqrt{z} = z^{1/2} = e^{\frac{1}{2}\log z}$ be defined on $im(f)$.

Then $\log z = \log|z| + i\arg z$ where $\arg z \in (0, 2\pi)$ by our branch cut in $im(f)$, thus we have

$$g(z) = e^{\frac{1}{2}\log|z|}e^{(\frac{1}{2}\arg z)i}$$

Since $\arg z \in (0, 2\pi)$, $\frac{1}{2}\arg z \in (0, \pi)$, hence $img \subseteq UHP$.

Now consider the map $T(z) = \frac{i-z}{i+z}$ sending UHP to D conformally; now we have



Hence $|T\circ g\circ f(z)| \leq 1$ for all $z \in \mathbb{C}$; furthermore, since f entire, g analytic on imf, and T analytic on img, we have $T\circ g\circ f$ is entire.

So by Liouville, $T\circ g\circ f \equiv$ constant

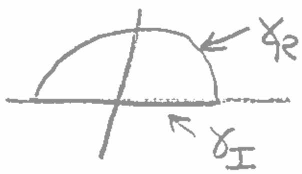
However, T and g were defined not to be constant, hence f must be constant.

③ $\int_0^\infty \frac{dx}{(1+x^2)\sqrt{x}}$

Let $x=u^2 \Rightarrow dx = 2u\,du \Rightarrow \int_0^\infty \frac{dx}{(1+x^2)\sqrt{x}} = \int_0^\infty \frac{2u\,du}{(1+u^4)u} = \int_0^\infty \frac{2du}{(1+u^4)}$

Now $\frac{1}{1+u^4}$ is even, hence $\int_0^\infty \frac{2du}{(1+u^4)} = \int_{-\infty}^\infty \frac{du}{1+u^4}$

Now consider the function $f(z) = \frac{1}{1+z^4}$ integrated over the contour $\vec{\Gamma}_R =$



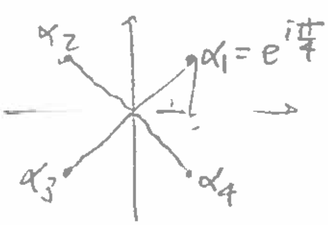
Note that $\lim_{R\to\infty} \int_{\gamma_I} \frac{dz}{1+z^4} = \int_{-\infty}^\infty \frac{du}{1+u^4}$; now see the following:

$$\left|\int_{\gamma_R} \frac{dz}{1+z^4}\right| \le \int_{\gamma_R} \frac{|dz|}{|1+z^4|} \le \int_{\gamma_R} \frac{|dz|}{|z|^4-1} = \frac{\pi R}{R^4-1} \longrightarrow 0 \text{ as } R\to\infty$$

So, in fact, $\lim_{R\to\infty} \int_{\vec{\Gamma}_R} \frac{dz}{1+z^4} = \lim_{R\to\infty}\left(\int_{\gamma_R} + \int_{\gamma_I}\right) = \lim_{R\to\infty} \int_{\gamma_I} = \int_{-\infty}^\infty \frac{du}{1+u^4}$

So now we will compute $\int_{\vec{\Gamma}_R} \frac{dz}{1+z^4}$ with the residue thm.

Note that $f(z)$ has 4 simple poles, 2 of which are in the UHP ; call those α_1, α_2 and the other two α_3, α_4 ; Then:



- $\operatorname{Res}_{z=\alpha_1} f(z) = \frac{1}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)(\alpha_1-\alpha_4)} = \frac{1}{(\sqrt{2})(2\alpha_1)(i\sqrt{2})}$

$$= \frac{1}{2i\left(2\left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)\right)} = \frac{1}{2i(\sqrt{2}+i\sqrt{2})} = \frac{1}{2\sqrt{2}(i-1)}$$

- $\operatorname{Res}_{z=\alpha_2} f(z) = \frac{1}{(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)(\alpha_2-\alpha_4)} = \frac{1}{(-\sqrt{2})(i\sqrt{2})(2\alpha_2)} = \frac{1}{-2\left(2i\left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)\right)}$

$$= \frac{1}{-2(-i\sqrt{2}-\sqrt{2})} = \frac{1}{2\sqrt{2}(i+1)}$$

Then

$$\int_{-\infty}^\infty \frac{dx}{(1+x^2)\sqrt{x}} = 2\pi i\left(\frac{1}{2\sqrt{2}(i+1)} + \frac{1}{2\sqrt{2}(i-1)}\right) = \frac{\pi i}{\sqrt{2}}\left(\frac{1}{i+1}+\frac{1}{i-1}\right) = \frac{\pi i}{\sqrt{2}}(-i) = \boxed{\frac{\pi}{\sqrt{2}}}$$

(4.) Show $\prod_{k=0}^{\infty}(1+z^{2^k})$ converges for $|z|<1$ and equals $\frac{1}{1-z}$

Consider the partial products $p_n(z) = \prod_{k=0}^{n}(1+z^{2^k})$

Then $(1-z)p_n(z) = \underbrace{\underbrace{\underbrace{(1-z)(1+z)}_{=(1-z^2)}(1+z^2)}_{=(1-z^4)}(1+z^4)}_{=(1-z^8)} \cdots (1+z^{2^n})$

hence $(1-z)p_n(z) = (1-z^{2^{n+1}})$

hence $\lim_{n\to\infty}(1-z)\prod_{k=0}^{n}(1+z^{2^k}) = \lim_{n\to\infty}(1-z)p_n(z) = \lim_{n\to\infty}(1-z^{2^{n+1}})$

$= 1$ since $|z|<1$

$\Rightarrow \lim_{n\to\infty}\prod_{k=0}^{n}(1+z^{2^k}) = \frac{1}{1-z}$

$\Rightarrow \prod_{k=0}^{\infty}(1+z^{2^k}) = \frac{1}{1-z}$

(5.) $f: D \to D$ $(D = \{|z|<1\})$ has 2 distinct fixed points

Show $f(z) = z$:

Let z_1, z_2 be the fixed points.

By the Riemann Mapping Theorem, $\exists \varphi: D \to D$ with $\varphi(z_1) = 0$ and biholomorphic.

Now consider the function $\varphi \circ f \circ \varphi^{-1}$; clearly it maps $D \to D$ and $\varphi \circ f \circ \varphi^{-1}(0) = \varphi \circ f(z_1) = \varphi(z_1) = 0$, hence we may apply the Schwarz Lemma.

→ Considering the point $\varphi(z_2)$, we see that

$|\varphi \circ f \circ \varphi^{-1}(\varphi(z_2))| = |\varphi \circ f(z_2)| = |\varphi(z_2)|$,

hence by Schwarz $\varphi \circ f \circ \varphi^{-1}$ is a rotation, i.e.

$\varphi \circ f \circ \varphi^{-1}(z) = cz$ for $|c| = 1$

Now note that $\varphi \circ f \circ \varphi^{-1}(z_2) = cz_2 \Rightarrow \varphi \circ f(z_2) = c\varphi(z_2)$

$\Rightarrow \varphi(z_2) = c\varphi(z_2) \Rightarrow c = 1$ since $\varphi(z_2) \neq 0$

Hence we have $\varphi \circ f \circ \varphi^{-1}(z) = z$

$\Rightarrow \varphi \circ f(z) = \varphi(z) \Rightarrow \underline{f(z) = z}$