COMPLEX ANALYSIS GRADUATE EXAM

SPRING 2008

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Suppose $a, b \geq 0$. Show that

$$\int_{-\infty}^{\infty} \frac{\cos ax - \cos bx}{x^2}\, dx = \pi(b-a).$$

Deduce that

$$\int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^2 dx = \pi.$$

2. Discuss the uniform convergence of the series

$$\sum_{n=1}^{\infty} \frac{z}{n(1+nz^2)}$$

on the set $E = \{z = x + iy : 0 < y < x\}$.

3. (a) Let $f(z) = u(x,y) + iv(x,y)$ be a holomorphic function on a connected open set D. Suppose $au(x,y) + bv(x,y) = c$ on D, where a, b and c are real constants which are not all zero. Show that f is constant on D.

(b) Let f and g be holomorphic functions in a connected open set D. Suppose that f and g have no zeros in D. Suppose also that there is a sequence a_n of points in D such that $a_n \to a \in D$ and $a_n \neq a$ for all n and such that

$$\frac{f'(a_n)}{f(a_n)} = \frac{g'(a_n)}{g(a_n)} \quad \text{for all } n.$$

Show that there is a constant c such that $f(z) = cg(z)$ in D.

4. Let $f(z)$ be a holomorphic function on the disc $|z| < 1$ and suppose that $f(0) = 0$. Show that the series

$$\sum_{n=1}^{\infty} f(z^n) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} a_k z^{nk}$$

converges uniformly in any compact subset of this disc.

$$= \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} a_k z^{nk}$$

$$f(z) + \sum_{n=2}^{\infty} f(z^n)$$

$$= \sum_{k=1}^{\infty} a_k \sum_{n=1}^{\infty} (z^k)^n$$

$$= \sum_{k=1}^{\infty} a_k \frac{1}{1-z^k}$$

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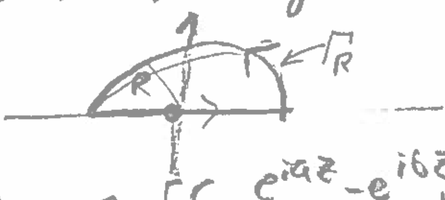
① Suppose $a, b \geq 0$. Show $\int_{-\infty}^{\infty} \frac{\cos ax - \cos bx}{x^2}\,dx = \pi(b-a)$

& deduce $\int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^2 dx = \pi$.

$\sin^2 x = \frac{1-\cos 2\theta}{2}$, here $= \frac{\pi(2-0)}{2} = \pi$

consider Γ given by:



and consider $\int_\Gamma \frac{e^{iaz} - e^{ibz}}{z^2}\,dz$

Then $\mathrm{Re}\left[\int_\Gamma \frac{e^{iaz}-e^{ibz}}{z^2}\,dz\right] = \int_\Gamma \frac{\cos az - \cos bz}{z^2}\,dz$

Now: $\int_\Gamma \frac{e^{iaz}-e^{ibz}}{z^2}\,dz = \pi i \left(\operatorname{Res}_{z=0} \frac{e^{iaz}-e^{ibz}}{z^2}\right)$

pole at zero contributes 1/2 residue since Γ passes thru it & integral form $\int_\Gamma e^{iz} R(z)\,dz$ rational w/ pole @ zero.

Now, $\frac{e^{iaz}-e^{ibz}}{z^2} = \frac{\left(1+iaz+\frac{1}{2}(iaz)^2+\cdots\right)-\left(1+ibz+\frac{1}{2}(ibz)^2+\cdots\right)}{z^2}$

$= 0 + \frac{iz(a-b)}{z^2} + \frac{\frac{1}{2}z^2(b^2-a^2)}{z^2} + \cdots$

$= 0 + \frac{i(a-b)}{z} + \frac{1}{2}(b^2-a^2) + \cdots$

hence $\operatorname{Res}_{z=0} \frac{e^{iaz}-e^{ibz}}{z^2} = i(a-b)$

Hence $\int_\Gamma \frac{e^{iaz}-e^{ibz}}{z^2}\,dz = \pi i(i(a-b)) = \pi(b-a)$

Now consider $\frac{e^{iaz}-e^{ibz}}{z^2}$ on $|z|=R$;

$y \geq 0$ in upper half plane.

$\left|\frac{e^{iaz}-e^{ibz}}{z^2}\right| \leq \frac{|e^{iaz}|+|e^{ibz}|}{R^2} = \frac{e^{-ay}+e^{-by}}{R^2} \leq \frac{2}{R^2} \to 0$ as $R \to \infty$.

hence $\int_{\Gamma_R} \frac{e^{iaz}-e^{ibz}}{z^2}\,dz \to 0$ as $R \to \infty$.

$\pi(b-a) = \mathrm{Re}(\pi(b-a))$

hence $\int_{-\infty}^{\infty} \frac{\cos az - \cos bz}{z^2}\,dz = \mathrm{Re}\left[\int_{-\infty}^{\infty} \frac{e^{iaz}-e^{ibz}}{z^2}\,dz\right] = \mathrm{Re}\left[\int_\Gamma \frac{e^{iaz}-e^{ibz}}{z^2}\,dz\right]$

② Discuss unif. conv. of $\sum_{n=1}^{\infty} \frac{z}{n(1+nz^2)}$ on $E = \{z = x+iy : 0 < y < x\}$.

$$\sum_{n=1}^{\infty} \frac{z}{n(1+nz^2)} = \sum_{n=1}^{\infty} \frac{z}{zn^{3/2}\left(\frac{1}{n^{1/2}z} + n^{1/2}z\right)}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^{3/2}\left(\frac{1}{zn^{1/2}} + zn^{1/2}\right)}$$

But $\left|\frac{1}{n^{1/2}z} + n^{1/2}z\right| \geq 1 \Rightarrow \frac{1}{\left|\frac{1}{n^{1/2}z} + n^{1/2}z\right|} \leq 1$

$$\Rightarrow \frac{1}{n^{3/2}\left(\frac{1}{n^{1/2}z} + n^{1/2}z\right)} \leq \frac{1}{n^{3/2}} = M_n$$

and $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}} < \infty$

by the p-test, hence

$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}\left(\frac{1}{zn^{1/2}} + zn^{1/2}\right)}$ unif conv. by weierstrass M-test.

④ $f(z)$ holomorphic fn on $|z|<1$ & suppose $f(0)=0$.
Show that $\sum_{n=1}^{\infty} f(z^n)$ converges unif. on any cpt. subsets of the disc.

$f(z)$ holomorphic $\Rightarrow f(z) = \sum_{k=0}^{\infty} a_k z^k$

$\Omega \subseteq D$ compact $\Rightarrow |z| \leq M < 1$ for all $z \in \Omega$ for some M.

Then: $|f(z^n)| = \left|\sum_{k=0}^{\infty} a_k z^{nk}\right| \leq \sum_{k=0}^{\infty} |a_k||z|^{nk} \leq \sum_{k=0}^{\infty} |a_k| M^{nk}$

$= \sum_{k=0}^{\infty} |a_k| (M^n)^k < \infty$

since $M^n < 1$ & f holomorphic on $|z|<1$.

Since $f(0)=0$, we have $a_0 = 0$.

Then: $\sum_{k=0}^{\infty} |a_k| (M^n)^k = |a_1| M^n + |a_2| M^{2n} + |a_3| M^{3n} + \cdots$

$= \sum_{k=1}^{\infty} |a_k| (M^n)^k = M^n \sum_{k=0}^{\infty} |a_{k+1}| (M^n)^k$