COMPLEX ANALYSIS GRADUATE EXAM
Fall 2008

Answer all five questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

(1) Map the region $\{|z| < 1\} \setminus \{|z - \frac{1}{2}| < \frac{1}{2}\}$ conformally to the upper half plane.

(2) Evaluate the integral

$$\int_0^\infty \frac{1}{x^{1/2}(1+x^2)}\, dx.$$

(3) Assume f is meromorphic in $\{|z| \le 1\}$, and $|f(z)| = 1$ for all z with $|z| = 1$. Show that f is a rational function.

(4) A *fixed point* of a mapping f is a point z such that $f(z) = z$. Let $G = (0,1)^2$ be the open until square in $\mathbb{C}$. Show that if a holomorphic map $f : G \to G$ has two distinct fixed points, then it is the identity mapping.

(5) Let f be analytic in the unit disc, satisfying

$$M = \int_{\{|z|<1\}} |f(z)|\, dx\, dy < \infty.$$

Show that for all $|z| < 1$,

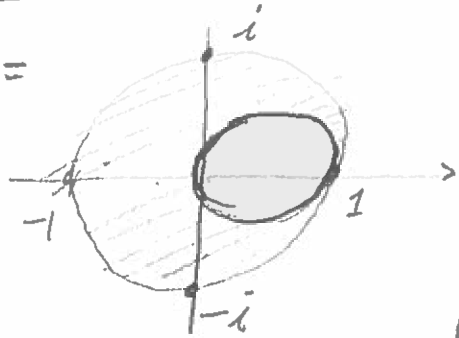
$$|f(z)| \le \frac{M}{\pi(1-|z|)^2}.$$

1. Conformal mapping
2. Residue integration
3. Reflection principle + Liouville thm.
4. Schwarz lemma
5. Cauchy integral formula + $dx\,dy = \rho\, d\rho\, d\theta$

CA – Fall '08

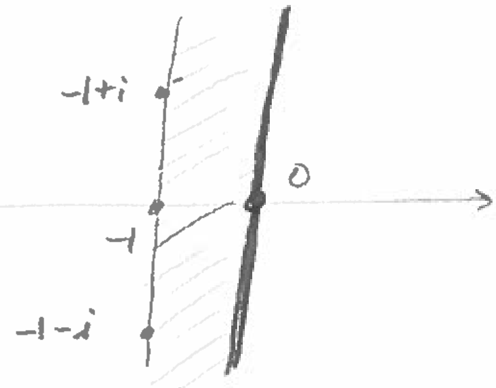
1. Conformally Map $\Omega = \{|z|<1\} \setminus \{|z-\tfrac{1}{2}|<\tfrac{1}{2}\}$ onto UHP

$\Omega =$

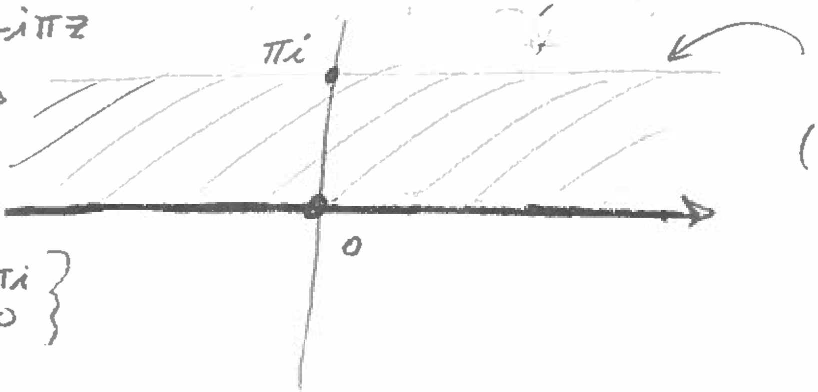


$f(z) = \dfrac{2z}{1-z}$ $\rightsquigarrow$

$$\left\{\begin{array}{l} 0 \mapsto 0 \\ 1 \mapsto \infty \\ -1 \mapsto -1 \\ i \mapsto -1+i \\ -i \mapsto -1-i \end{array}\right\}$$



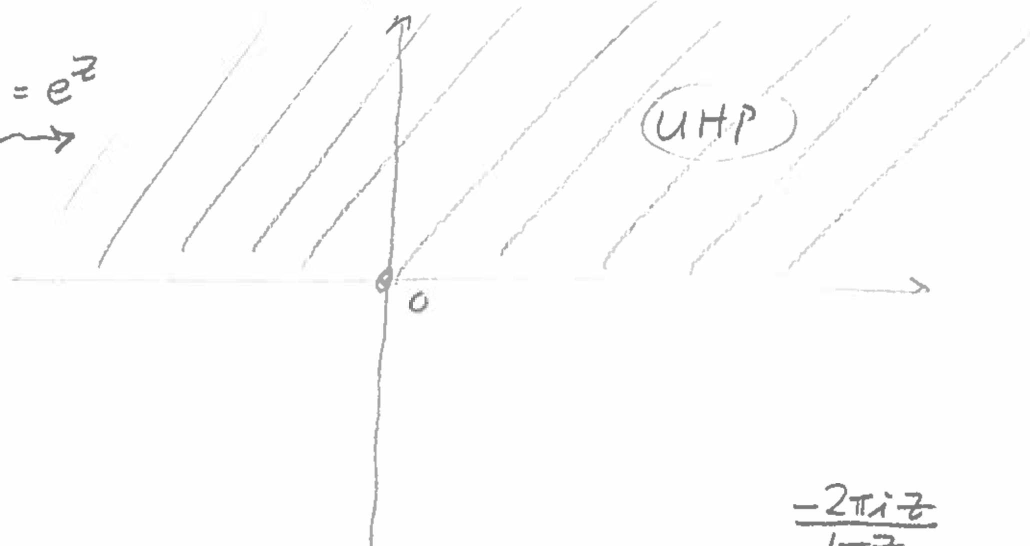
$g(z) = -i\pi z$ $\rightsquigarrow$



$$\left\{\begin{array}{l} -1 \mapsto \pi i \\ 0 \mapsto 0 \end{array}\right\}$$

(rotate $\frac{\pi}{2}$ clockwise)

$h(z) = e^{z}$ $\rightsquigarrow$



hence our map is $h \circ g \circ f(z) = e^{\frac{-2\pi i z}{1-z}}$

(2.) Evaluate $\int_0^\infty \frac{1}{x^{1/2}(1+x^2)}\,dx$

Let $u^2=x$, hence $2u\,du=dx$, and we get:

$$\int_0^\infty \frac{1}{x^{1/2}(1+x^2)}dx = \int_0^\infty \frac{2u\,du}{u(1+u^4)} = \int_0^\infty \frac{2\,du}{(1+u^4)} = 2\int_0^\infty \frac{du}{(1+u^4)}$$

Now consider the function

$f(z) = \frac{1}{1+z^4}$ and consider the integral over Γ_R:

Note now that, as $R\to\infty$,

$$\int_{\gamma_{real}} f(z)dz = \int_{-\infty}^{\infty}\frac{dx}{1+x^4} = 2\int_0^\infty \frac{dx}{1+x^4}$$ since integrand is even function.

Now consider the integral over γ_R:

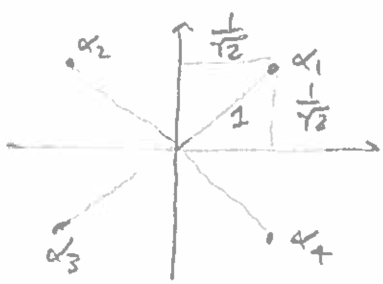
$$\left|\int_{\gamma_R} f(z)dz\right| \le \int_{\gamma_R} |f(z)||dz| = \int_{\gamma_R}\frac{|dz|}{|1+z^4|} \le \frac{1}{R^4-1}\int_{\gamma_R}|dz| = \frac{\pi R}{R^4-1} \to 0 \text{ as } R\to\infty$$

since $|z^4+1| \ge |z^4|-|-1|| = |z|^4-1 = R^4-1$

Hence $\int_{\Gamma_R} f(z)dz = \int_0^\infty \frac{2\,dx}{1+x^4}$ as $R\to\infty$.

Now apply the Residue Theorem, namely; for the 2 poles α_1, α_2 of f in the UHP (and hence the two inside Γ_R as $R\to\infty$), we have:

$$\int_{\Gamma_R} f(z)dz = 2\pi i\left(\operatorname{Res}_{z=\alpha_1} f(z) + \operatorname{Res}_{z=\alpha_2} f(z)\right),$$ so compute:



$$\operatorname{Res}_{z=\alpha_1} f(z) = \frac{1}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)(\alpha_1-\alpha_4)} = \frac{1}{(\sqrt2)(2\alpha_1)(i\sqrt2)} = \frac{1}{2\sqrt2(i-1)}$$

$$\operatorname{Res}_{z=\alpha_2} f(z) = \frac{1}{(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)(\alpha_2-\alpha_4)} = \frac{1}{(-\sqrt2)(i\sqrt2)(2\alpha_2)} = \frac{1}{2\sqrt2(i+1)}$$

Then

$$2\pi i\left(\frac{1}{2\sqrt2}\left(\frac{1}{i-1}+\frac{1}{i+1}\right)\right) = \frac{\pi i}{\sqrt2}\left(\frac{i+1+i-1}{(i-1)(i+1)}\right) = \frac{\pi i}{\sqrt2}(-i) = \frac{\pi}{\sqrt2}$$

(3) f meromorphic in $|z|<1$ and $|f(z)|=1$ for all $|z|=1$

Show f is rational!

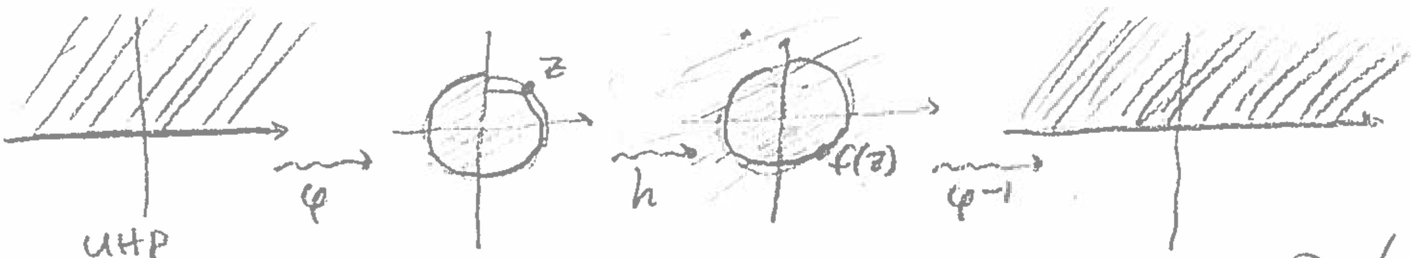
Let $\{a_1,\dots,a_k\}$ be the zeroes of f with orders n_i, and let $\{p_1,\dots,p_\ell\}$ be the poles of f with orders m_i.

Now define $P_i(z) = \frac{a_i - z}{1-\bar{a}_i z}$, $Q_i(z) = \frac{p_i - z}{1-\bar{p}_i z}$, and let $g(z) = \frac{\prod P_i(z)^{n_i}}{\prod Q_i(z)^{m_i}}$

and let $h(z) = \frac{f(z)}{g(z)}$.

Now by construction, $|h(z)|=1$ for $|z|=1$ and h is analytic in $|z|<1$, and h is non-zero in $|z|<1$.

Now consider the map $\varphi(z) = \frac{i-z}{i+z}$. Then



Hence $\varphi^{-1}\circ h\circ\varphi$ is analytic on UHP, which sends $\mathbb{R}$ to $\mathbb{R}$, hence we may apply the Reflection Principle, i.e. we may extend $\varphi^{-1}\circ h\circ\varphi$ to an entire function via $\varphi^{-1}\circ h\circ\varphi(\bar{z}) = \overline{\varphi^{-1}\circ h\circ\varphi(z)}$

Now note that since $\varphi^{-1}\circ h\circ\varphi$ is entire, so must h be entire, and we have $h = \varphi\circ f\circ\varphi^{-1}$,

note that since $\varphi^{-1}\circ h\circ\varphi = f$, f entire,

hence h is bounded.

So h bounded + entire, so constant by Liouville.

Then $\frac{f}{g} = \text{const} \Rightarrow f = \text{const}\cdot g$, a rational function since g was rational

④ $G = (0,1) \times (0,1) \subseteq \mathbb{C}$; show that if $f: G \to G$ holomorphic with 2 fixed points, then f is identity.

By the Riemann Mapping Theorem, we have a bijective, biholomorphic map $\varphi: G \to D$ s.t. $\varphi(z_1) = 0$ (since G clearly simply-connected)

Now $\varphi \circ f \circ \varphi^{-1}: D \to D$ and $\varphi \circ f \circ \varphi^{-1}(0) = \varphi \circ f(z_1) = \varphi(z_1) = 0$, hence by Schwarz lemma, $|\varphi \circ f \circ \varphi^{-1}(z)| \leq |z|$.

Now consider the point $\varphi(z_2)$ and see that:

$|\varphi \circ f \circ \varphi^{-1}(\varphi(z_2))| = |\varphi \circ f(z_2)| = |\varphi(z_2)|$,

hence $\varphi \circ f \circ \varphi^{-1}(z) = cz$, $|c| = 1$ (i.e. a rotation), again by Schwarz.

Now consider z_2: $\varphi \circ f \circ \varphi^{-1}(z_2) = cz_2 \Rightarrow \varphi \circ f(z_2) = c\varphi(z_2)$

$\Rightarrow z_2 = f(z_2) = \varphi^{-1}(c\varphi(z_2))$

$\Rightarrow \varphi(z_2) = c\varphi(z_2) \Rightarrow c = 1$ since $\varphi(z_2) \neq 0$

Therefore $\varphi \circ f \circ \varphi^{-1}(z) = z$

$\Rightarrow \varphi \circ f(z) = \varphi(z) \Rightarrow \underline{f(z) = z}$, the identity map