COMPLEX ANALYSIS GRADUATE EXAM

SPRING 2007

Answer all five questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper. D denotes the open unit disc $\{z \in \mathbb{C} : |z| < 1\}$ in the complex plane.

Note that some problems are worth 10 points and others are worth 7 points.

FToC + Cauchy Riem.

1. (10 points) Let $\Omega \subset \mathbb{C}$ be a convex domain and let $f : \Omega \to \mathbb{C}$ be a nonconstant holomorphic function satisfying $\mathrm{Re}(f'(z)) \geq 0$ for all $z \in \Omega$. Prove that f is injective on Ω.

Schwarz Lemma

2. (10 points) Let f be an analytic function on D satisfying $|f(z)| \leq 1$ for all $z \in D$ and having at least two fixed points z_1 and z_2. Show that $f(z) = z$ for all $z \in D$.

Conformal mapping

3. (10 points) Find a conformal mapping of the semicircular region $R = \{z : \mathrm{Im}(z) > 0, |z| < 1\}$ onto D. HINT: You may decompose this map into simpler ones.

4. (7 points) Evaluate the integral

Residue

$$\int_{-\infty}^{\infty} \frac{e^{-ix}}{x^2 - 2x + 4}\, dx.$$

Justify your method.

Rouché

5. (7 points) How many roots does the polynomial $p(z) = 2z^5 + 4z^2 + 1$ have in D? Justify your answer.

Complex Analysis: Spring 2007

1. Let $\Omega \subseteq \mathbb{C}$ convex domain & let $f: \Omega \to \mathbb{C}$ nonconst, holomorphic fn, satisfying $Re(f'(z)) \geq 0$ for all $z \in \Omega$.
Prove f is injective on Ω.

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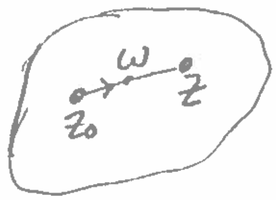
Ω convex:



So $f(z) - f(z_0) = \int_{z_0}^{z} f'(w)\,dw$, and

If f is not injective, $\int_{z_0}^{z} f'(w)\,dw = 0$ for some $z \neq z_0$.

Now, $\int_{z_0}^{z} Re(f'(w))\,dw + i \int_{z_0}^{z} Im(f'(w))\,dw = 0$

$\Rightarrow Re(f'(w)) = 0$ for all $w \in [z_0, z]$ straight line pictured

since $Re(f'(w)) \geq 0$, but $f'(w)$ is analytic,

hence $\int$ is path-independent, hence $f'(w) \equiv 0$, i.e.

$f(w) = $ const.

$\Rightarrow f(z) = $ const $+ i v(x,y)$

and hence $v(x,y) = $ const. by Cauchy-Riemann,

hence $f \equiv$ const, a contradiction.

Therefore f is injective.

② f analytic on D and $|f(z)| \leq 1$ $\forall z \in D$ and f has two fixed points z_1 & z_2. Show $f(z) = z$ $\forall z \in D$. (D = open unit disk)

So $f: D \to \overline{D}$ and $\begin{cases} f(z_1) = z_1 \\ f(z_2) = z_2 \end{cases}$

Let $T: D \to D$ be given by $T(z) = \dfrac{z - z_1}{1 - \overline{z}_1 z}$, hence

$T(z_1) = 0$ and $|T(z)| \leq 1$, hence:

$|T \circ f \circ T^{-1}(z)| \leq 1$ and $T \circ f \circ T^{-1}(0) = 0$

So apply Schwarz, hence $|T \circ f \circ T^{-1}(z)| \leq |z|$ for all $z \in D$.

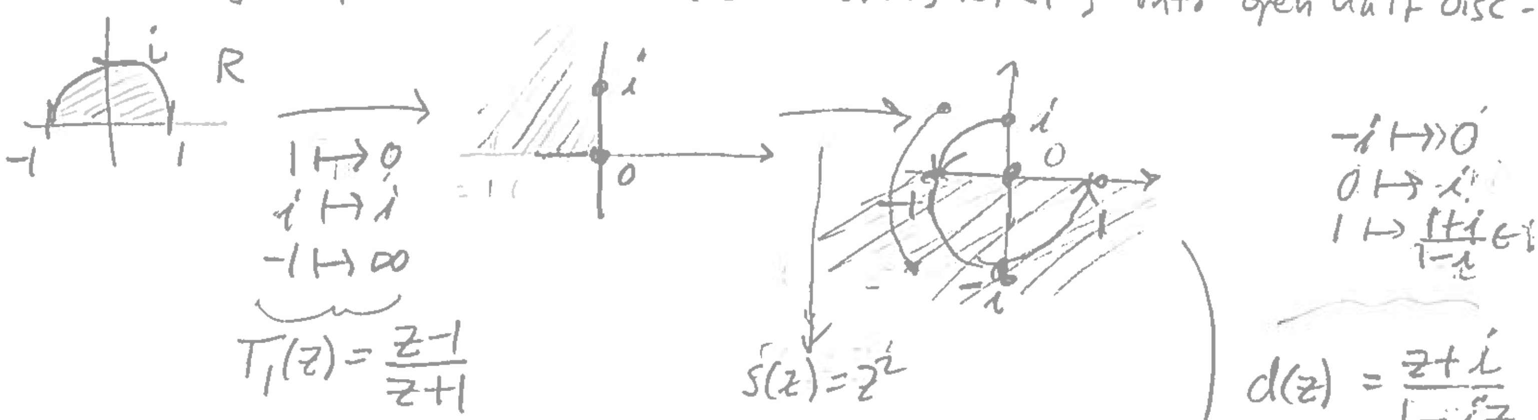
Let $a = T(z_2)$ and then:

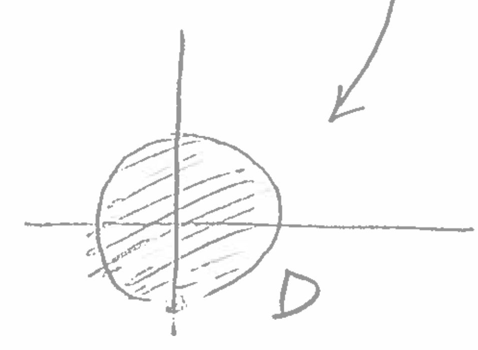
$$|T \circ f \circ T^{-1}(a)| = |T \circ f \circ T^{-1}(T(z_2))| = |T \circ f(z_2)| = |T(z_2)| = |a|,$$

hence (also by Schwarz), $f(z) = cz$ for $|c| = 1$,

but f has two nonzero fixed pts, hence this rotation must be $0°$, hence $\underline{f(z) = z}$.

③ Conformally map semicircle $R = \{z : \operatorname{Im}(z) > 0, |z| < 1\}$ onto open unit disc.





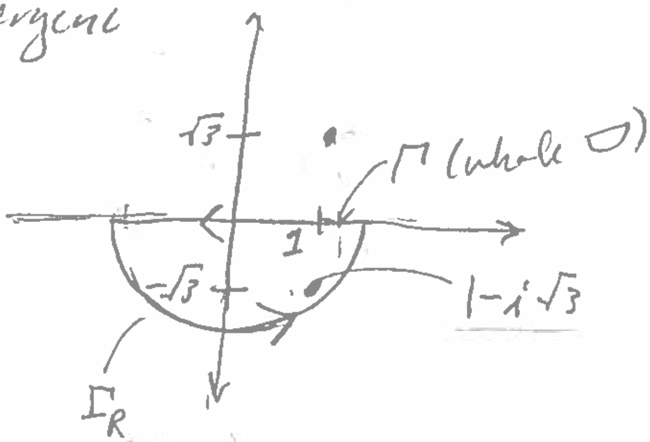
So the conformal map is

$\underline{d \circ s \circ T_1}$

④ Evaluate $\int_{-\infty}^{\infty} \frac{e^{-ix}}{x^2-2x+4}\,dx$

See that the roots of the denominator are $\frac{2\pm\sqrt{4-4^2}}{2} = \frac{2\pm\sqrt{-12}}{2} = 1\pm i\sqrt{3}$

(Recall that $|e^{-i(x+iy)}| = e^y$, so use the lower half plane so convergent)



Let $f(z) = \frac{e^{-iz}}{z^2-2z+4}$

Now $\int_\Gamma f(z)\,dz = 2\pi i\left(\operatorname{Res}_{z=1-i\sqrt{3}} f(z)\right)$

$$= 2\pi i\left(\lim_{z\to 1-i\sqrt{3}} f(z)(z-(1-i\sqrt{3}))\right)$$

$$= 2\pi i\left(\left.\frac{e^{-iz}}{z-(1+i\sqrt{3})}\right|_{z=1-i\sqrt{3}}\right)$$

$$= 2\pi i\left(\frac{e^{-i(1-i\sqrt{3})}}{1-i\sqrt{3}-(1+i\sqrt{3})}\right) = \frac{2\pi i\, e^{-(i+\sqrt{3})}}{-2i\sqrt{3}}$$

$$= \frac{-\pi e^{-(1+\sqrt{3})}}{i\sqrt{3}}$$

so $= \frac{\pi e^{-(i+\sqrt{3})}}{i\sqrt{3}}$

(change sign since contour integral goes $-R\to R$)

Now, $\int_\Gamma = \int_{-R}^{R} + \int_{\Gamma_R}$

$$\left|\int_{\Gamma_R}\frac{e^{-iz}}{z^2-2z+4}dz\right| \le \int_{\Gamma_R}\frac{|e^{-iz}|}{|z^2-2z+4|}|dz| = \int_{\Gamma_R}\frac{e^y}{|z^2-2z+4|}|dz| \overset{\text{since } y\le 0}{\le} \int_{\Gamma_R}\frac{1}{|z^2-2z+4|}|dz|$$

and see that $|(z-\underbrace{(1-i\sqrt{3})}_{\text{modulus } 2})\cdot(z-\underbrace{(1+i\sqrt{3})}_{\text{modulus } 2})| \ge |R-2||R-2| = (R-2)^2$
$= R^4+4R+4$

hence $\int_{\Gamma_R}\frac{|dz|}{|z^2-2z+4|} \le \frac{1}{R^4+4R+4}\int_{\Gamma_R}|dz| = \frac{\pi R}{R^2-4R+4} \to 0$ as $R\to\infty$

(5) How many roots does $p(z) = 2z^5 + 4z^2 + 1$ have in unit disc?

~~Argument principle:~~ $\cancel{\int_{\partial D} \frac{p'(z)}{p(z)} dz = 2\pi i (N)}$ ~~# zeros inside~~ $\cancel{\gamma}$.

$$\cancel{\int_{\partial D} \frac{10z^4+8z}{2z^5+4z^2+1} dz = \int_0^{2\pi} \frac{10e^{4i\theta}+8e^{i\theta}}{2e^{5i\theta}+4e^{2i\theta}+1} ie^{i\theta} d\theta}$$

$$\cancel{= i\int_0^{2\pi} \frac{10e^{5i\theta}+8e^{2i\theta}}{2e^{5i\theta}+4e^{2i\theta}+1} d\theta}$$

Apply Rouché:

Let $f(z) = 4z^2$. Then for $|z| = 1$,

$$|p(z) - f(z)| = |2z^5 + 1| \leq 2|z|^5 + 1 = 3$$
$$|f(z)| = |4z^2| = 4|z|^2 = 4$$

$\Rightarrow$ $|p(z) - f(z)| \leq |f(z)|$ on ∂D,

hence they have the same # of zeros on D; note that $f(z) = 4z^2$ has one zero of order 2 at 0, hence

$p(z)$ has 2 zeros in D