COMPLEX ANALYSIS GRADUATE EXAM

FALL 2007

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Suppose $a > 1$. Show that

Residue

$$\int_0^{2\pi} \frac{d\theta}{a + \sin\theta} = \frac{2\pi}{\sqrt{a^2 - 1}}.$$

Cauchy est.

2. Suppose that $f : \mathbb{C} \to \mathbb{C}$ is entire. Show that the Taylor series of f at 0 converges to f uniformly on $\mathbb{C}$ if and only if f is a polynomial.

Schwarz Lemma

3. Let f be a one-to-one holomorphic function on the unit disc $B_1 = \{z \in \mathbb{C} : |z| < 1\}$ and let $D = f(B_1)$ be the image of B_1 under f. Similarly let D_r be the image under f of the open disc $B_r = \{z \in \mathbb{C} : |z| < r\}$ for $0 < r < 1$. Show that if $h : D \to D$ is a holomorphic mapping leaving the point $f(0)$ fixed then

$$h(D_r) \subseteq D_r \text{ for } 0 < r < 1.$$

Reflection

4. Let $f(z)$ be continuous in $\operatorname{Re} z \geq 0$ and analytic in $\operatorname{Re} z > 0$. Let $g(x)$ be continuous in $\operatorname{Re} z \leq 0$ and analytic in $\operatorname{Re} z < 0$. Assume that $f = g$ on $\operatorname{Re} z = 0$. Prove that f and g are differentiable in the closure of their domains and that $\partial f/\partial x = \partial g/\partial x$ at the origin.

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④

① $a>1$. Show: $\int_0^{2\pi} \frac{d\theta}{a+\sin\theta} = \frac{2\pi}{\sqrt{a^2-1}}$

Make the substitution $z=e^{i\theta}$, hence $\cos\theta = \frac{1}{2}(z+\frac{1}{z})$ & $\sin\theta = \frac{1}{2i}(z-\frac{1}{z})$ & $d\theta = \frac{dz}{iz}$.

$$\int_0^{2\pi} \frac{d\theta}{a+\sin\theta} = -i\int_{|z|=1} \frac{1}{a+\frac{1}{2i}(z-\frac{1}{z})}\cdot\frac{dz}{z}$$

$$= -i\int_{|z|=1} \frac{dz}{az+\frac{z^2}{2i}-\frac{1}{2i}} = -i\int_{|z|=1} \frac{2i\,dz}{z^2+2iaz-1}$$

Now see that this rational fn has poles at $z = \frac{-2ia \pm \sqrt{-4a^2+4(1)(+1)}}{2}$

$$= \frac{-2ia\pm\sqrt{4-4a^2}}{2} = \frac{-2ia \pm 2\sqrt{1-a^2}}{2} = -ia \pm \sqrt{1-a^2}$$

$= -ia \pm i\sqrt{a^2-1} = i(-a \pm \sqrt{a^2-1})$, hence on the imaginary axis.

Now: $|i(-a-\sqrt{a^2-1})| = a+\sqrt{a^2-1} > 1$ and

$|i(-a+\sqrt{a^2-1})| = a-\sqrt{a^2-1} < 1$ since $(a+\sqrt{a^2-1})(a-\sqrt{a^2-1})$
$= a^2-(a^2-1) = 1$

Therefore must only consider residue at

$z = i(-a+\sqrt{a^2-1}) = \alpha$

hence $-i\int_{|z|=1} \frac{2i\,dz}{z^2+2iaz-1} = \int_{|z|=1} \underbrace{\frac{2\,dz}{z^2+2iaz-1}}_{f(z)} = 2\pi i\left(\operatorname{Res}_{z=\alpha} f(z)\right)$

α is a simple pole, so $\operatorname{Res}_{z=\alpha} f(z) = \lim_{z\to\alpha}(z-\alpha)f(z)$

$$= \lim_{z\to\alpha} \frac{2(z-\alpha)}{z^2+2iaz-1}$$

Let $\beta = i(-a-\sqrt{a^2-1})$

$$= \lim_{z\to\alpha} \frac{2}{(z-\beta)} = \frac{2}{(\alpha-\beta)}$$

So $= 2\pi i\left(\frac{2}{\alpha-\beta}\right) = \frac{4\pi i}{i(-a+\sqrt{a^2-1}+(+a+\sqrt{a^2-1}))} = \frac{4\pi}{2\sqrt{a^2-1}} = \frac{2\pi}{\sqrt{a^2-1}}$

② $f: \mathbb{C} \to \mathbb{C}$ entire. Show that the Taylor series of f at 0 converges to f unif on $\mathbb{C}$ $\Leftrightarrow$ f polynomial.

($\Leftarrow$) Clear; f polynomial $\Rightarrow f(z) = \sum_{i=0}^{\infty} a_i z^i$ and $a_i = 0$ for all $i > \deg f = n$.
(then for any $\varepsilon > 0$, for all $m > n$ we get $f_m = \sum_{i=0}^{m} a_i z^i$ and $|f - f_m| = 0 < \varepsilon$)

($\Rightarrow$) Suppose f is not polynomial, i.e. $\forall m$, $\exists N > m$ s.t. $a_N \neq 0$.

Let $f(z) = \sum_{i=0}^{\infty} a_i z^i$ and let $T_n(z) = \sum_{i=0}^{n} a_i z^i$.

Then $|T_m(z) - f(z)| = \left| \sum_{n=m+1}^{\infty} a_n z^n \right| = \left| \sum_{n=N}^{\infty} a_n z^n \right|$

Now suppose $\varepsilon > 0$ and $\exists M$ such that

$$|T_m(z) - f(z)| < \varepsilon \quad \text{for all } z \in \mathbb{C},\ m > M$$

Then $|f(z)| \leq |T_m(z)| + \varepsilon \leq C(1 + |z|^m) + \varepsilon$

since polynomial

and recall Cauchy est: $|f^{(n)}(0)| \leq \dfrac{n!\, M(r)}{r^n} \leq \dfrac{n!\,(C(1+r^m)+\varepsilon)}{r^n}$

$\to 0$ as $r \to \infty$

(f entire)

therefore $|f^{(n)}(0)| = 0$
for all $n > M$, i.e. $a_n = 0$ for all $n > M$,
hence f is a polynomial.

(3) Let $f: B_1 \to B_1$ be one to one, holomorphic on $B_1 = \{z \in \mathbb{C} : |z| < 1\}$ and let $D = f(B_1)$ be the image.

Let $B_r = \{z \in \mathbb{C} : |z| < r\}$, $r < 1$, and $D_r = f(B_r)$

Show that if $h: D \to D$ holomorphic $\ni h(f(0)) = f(0)$ then:

$h(D_r) \subseteq D_r$ for $0 < r < 1$.

Utilize Schwarz Lemma; f^{-1} exists on $D = f(B_1)$ since 1-1.

Consider $f^{-1} \circ h \circ f : B_1 \to B_1$ which is analytic in $B_1 = \{z \in \mathbb{C} : |z| < 1\}$ $\ni$ $f^{-1} \circ h \circ f(0) = 0$, hence by Schwarz lemma we have $|f^{-1} \circ h \circ f(z)| \leq |z|$

Hence for $z \in B_r$, $|f^{-1} \circ h \circ f(z)| \leq |z| < r$,

hence $f^{-1} \circ h \circ f(B_r) \subseteq B_r$

$\Rightarrow f^{-1} \circ h(D_r) \subseteq B_r$

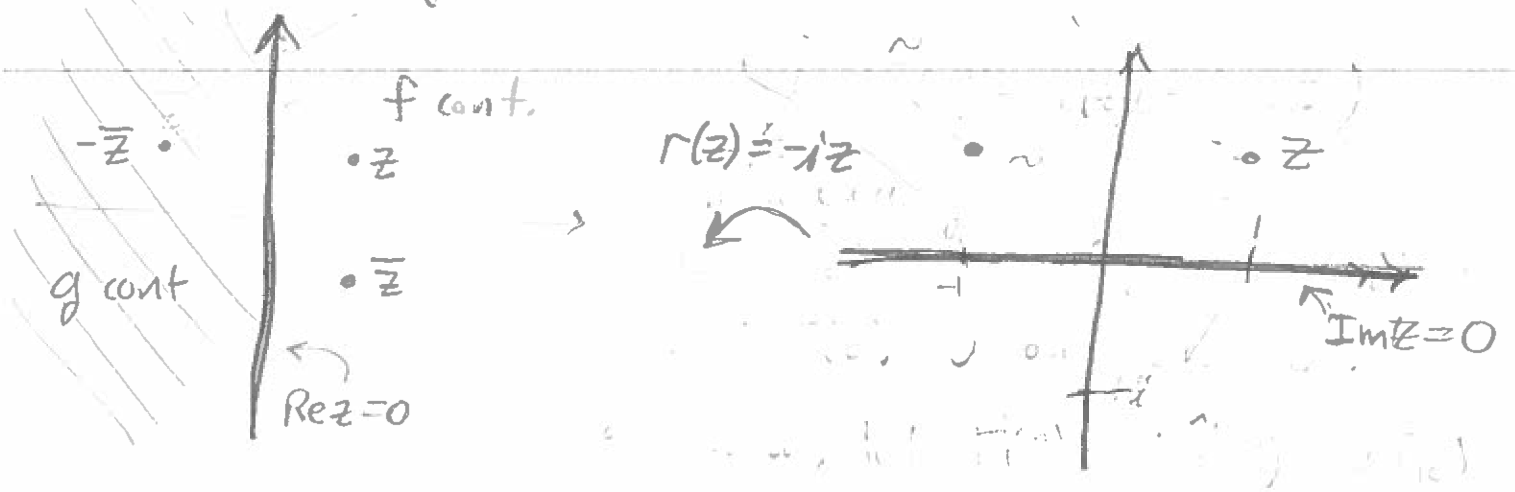
$\Rightarrow h(D_r) \subseteq f(B_r) = D_r$.

$\Rightarrow$ $\underline{h(D_r) \subseteq D_r}$

③ f cont in $\operatorname{Re} z \geq 0$ & analytic in $\operatorname{Re} z > 0$, g cont in $\operatorname{Re} z \leq 0$ & analytic in $\operatorname{Re} z < 0$

Assume $f = g$ on $\operatorname{Re} z = 0$ & prove that f & g are differentiable in the closure of their domains and that $f_x = g_x$ at origin.

Consider the diagram



Let $\tilde{F}(z) = f(z) - g(-\bar{z})$ and let

$$F(z) = \tilde{F} \circ r(z) = \tilde{F}(-iz) = f(-iz) - g(-(\overline{-iz}))$$
$$= f(-iz) - g(-(i)(\bar{z}))$$
$$= f(-iz) - g(-i\bar{z})$$

So for $z \in \{\operatorname{Im}(z) \geq 0\}$, $F(z)$ is continuous and for $z \in \{\operatorname{Im}(z) > 0\}$, $F(z)$ is analytic, and clearly $F(z) = 0$ for all $z \in \{\operatorname{Im}(z) = 0\}$.

Therefore $\exists$ extension of F to whole plane such that $F(z) = \overline{F(\bar{z})}$ and analytic.