# COMPLEX ANALYSIS GRADUATE EXAM

## SPRING 2006

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{x^2}{x^4+1}\,dx,$$

being careful to justify your methods.

2. Let $G$ be the family of all analytic mappings from the half space $\{z : \operatorname{Re} z > 0\}$ into the open unit disc $\{z : |z| < 1\}$. Define $\theta = \sup\{|f'(\pi)| : f \in G\}$.
   (i) Find an upper bound for $\theta$.
   (ii) Show that there exists $g \in G$ such that $|g'(\pi)| = \theta$.

3. Suppose $f$ is analytic on $0 < |z| < \rho$.
   (i) Give the definitions of the three possible types of isolated singularity of $f$ at 0.
   (ii) Prove that $e^f$ cannot have a pole at 0.

4. Let $B(r)$ denote the open disc of radius $r$ centered at 0, and let $f : B(1) \to B(1)$ be a holomorphic map.
   (i) Assume that $f(0) \in (-1, 0)$. Show that $1/2 \notin f(B(1/2))$.
   (ii) Assume instead that $f(B(1/2)) \supset B(1/2)$. Show that there exists $\theta \in \mathbb{R}$ such that $f(z) = e^{i\theta}z$ for all $z \in B(1)$.

1. Residue integration

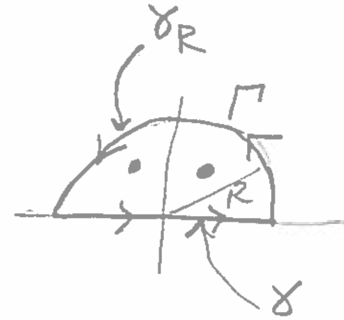
2. i. Cauchy estimate
   ii. Mobius transformations

3. definitions

4. Schwarz lemma

# CA – Spring 06:

① $\int_{-\infty}^{\infty} \frac{x^2}{x^4+1}\,dx$




Residue integration. Consider the contour $\Gamma$:
It will contain two of the poles of $\frac{z^2}{z^4+1}$, namely $z = e^{\frac{\pi i}{4}}$ and $z = e^{\frac{3\pi i}{4}}$

Then: $\int_{\Gamma} \frac{z^2}{z^4+1}\,dz = 2\pi i \left( \underset{z=e^{i\pi/4}}{\operatorname{Res}} f + \underset{z=e^{3\pi i/4}}{\operatorname{Res}} f \right)$

But also see that $\int_\Gamma = \int_{\gamma_R} + \int_\gamma$ and $\int_\gamma = \int_{-\infty}^{\infty} \frac{x^2}{x^4+1}\,dx$

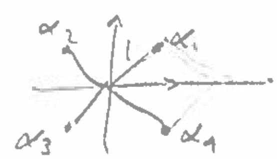
So now consider $\int_{\gamma_R}$:

$$\left| \int_{\gamma_R} \frac{z^2}{z^4+1}\,dz \right| \le \int_{\gamma_R} \frac{|z|^2}{|z^4+1|}\,|dz| \le \int_{\gamma_R} \frac{R^2}{R^4-1}\,|dz| = \frac{R^2}{R^4-1}(R\pi)$$

[Note that $|z^4+1| = |z^4-(-1)| \ge |z^4| - |-1| = R^4 - 1$
$\Rightarrow \frac{1}{|z^4+1|} \le \frac{1}{R^4-1}$ ]

Then since $\frac{\pi R^3}{R^4-1} \to 0$ as $R \to \infty$, we have $\int_{\gamma_R} \to 0$ as $R \to \infty$,

hence $\int_{-\infty}^{\infty} \frac{x^2}{x^4+1}\,dx = 2\pi i \left( \underset{z=e^{i\pi/4}}{\operatorname{Res}} f + \underset{z=e^{3\pi i/4}}{\operatorname{Res}} f \right)$



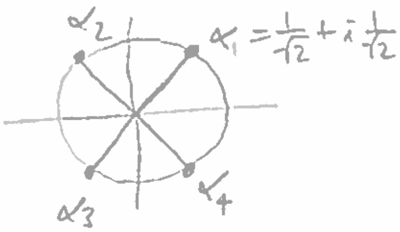

Now: $\underset{z=e^{i\pi/4}}{\operatorname{Res}} f = \lim_{z\to e^{\frac{\pi i}{4}}} \frac{(z-e^{\frac{\pi i}{4}})\, z^2}{z^4+1} = \lim_{z \to e^{\frac{\pi i}{4}}} \frac{z^2}{(z-e^{\frac{3\pi i}{4}})(z-\alpha_3)(z-\alpha_4)}$

$$= \frac{\alpha_1^2}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)(\alpha_1-\alpha_4)} = \frac{i}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)(\alpha_1-\alpha_4)} = \frac{i}{(\sqrt{2})(2\alpha_1)(i\sqrt{2})}$$

$$= \frac{1}{4\alpha_1} = \frac{1}{4(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}})} = \frac{1}{\frac{4}{\sqrt{2}} + i\frac{4}{\sqrt{2}}} = \frac{1}{2\sqrt{2} + i2\sqrt{2}} = \frac{1}{2\sqrt{2}(1+i)}$$

$$= \frac{(1-i)}{2\sqrt{2}(1+i)(1-i)} = \frac{1-i}{2\sqrt{2}(1^2-i^2)} = \frac{1-i}{4\sqrt{2}} = \boxed{\frac{\sqrt{2}(1-i)}{8}}$$

$$\operatorname*{Res}_{z=e^{3\pi i/4}} f = \lim_{z\to \underset{\alpha_2}{\underset{\shortparallel}{e^{3\pi i/4}}}} \frac{(z-e^{3\pi i/4})\,z^2}{z^4+1} = \frac{\alpha_2^{\,2}}{(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)(\alpha_2-\alpha_4)}$$




$$= \frac{\alpha_2^{\cancel{2}}}{(-\sqrt2)(i\sqrt2)(2\alpha_2)}$$

$$= \frac{\alpha_2}{-4i} = \frac{\alpha_2 i}{4}$$

$$= \frac{\left(-\frac{1}{\sqrt2} + i\frac{1}{\sqrt2}\right) i}{4}$$

$$= \left(-\frac{1}{\sqrt2} - i\frac{1}{\sqrt2}\right)\frac{1}{4}$$

$$= \frac{-1-i}{4\sqrt2} = \frac{-\sqrt2(1+i)}{8}$$

$$\alpha_2-\alpha_1 = \alpha_2+\alpha_3 = -\sqrt2$$
$$\alpha_2-\alpha_3 = \alpha_2+\alpha_1 = i\sqrt2$$
$$\alpha_2-\alpha_4 = \alpha_2+\alpha_2 = 2\alpha_2$$

So then $$\int_{-\infty}^{\infty} = 2\pi i\left(\frac{-\sqrt2(1+i)}{8} + \frac{\sqrt2(1-i)}{8}\right)$$

$$= 2\pi i\left(\frac{-\cancel{\sqrt2} - i\sqrt2 + \cancel{\sqrt2} - i\sqrt2}{8}\right)$$

$$= 2\pi i\left(\frac{-2i\sqrt2}{8}\right) = \frac{4\pi\sqrt2}{8} = \frac{\pi\sqrt2}{2} = \frac{\pi}{\sqrt2}$$

(2.) $G = \{ f : \{Re(z) > 0\} \to \{|z|<1\} \text{ analytic} \}$

Define $\Theta = \sup \{ |f'(\pi)| : f \in G \}$

(i) Find upper bound for $\Theta$

(ii) Show $\exists\, g \in G$ s.t. $|g'(\pi)| = \Theta$.

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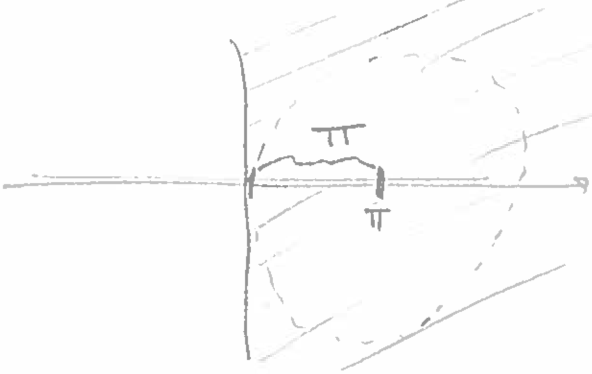
(i) Recall the Cauchy estimate: $|f^{(n)}(\pi)| \leq \dfrac{M(r)\, n!}{r^n}$

where $M(r) = \max\limits_{|z-\pi|=r} |f(z)|$, here $|f'(\pi)| \leq \dfrac{M(r)}{r}$

First, note that since $f$ maps into $D$, we have that $|f(z)|<1$ for all $z$, hence $M(r) \leq 1$ for all radii $r$.

On the other hand, note that since the domain of $f$ is the right half-plane, the largest radius a circle centred at $\pi$ may have is $\pi$.




Therefore we have obtained the bound:

$$|f'(\pi)| \leq \frac{M(\pi)}{\pi} \leq \frac{1}{\pi}$$

hence

$$\sup \{ |f'(\pi)| : f \in G \} \leq \frac{1}{\pi}$$

(ii) Let $g_a(z) = \dfrac{z-a}{z+\bar{a}}$

Then $g_a'(z) = \dfrac{(z+\bar{a})-(z-a)}{(z+\bar{a})^2} = \dfrac{\bar{a}+a}{(z+\bar{a})^2} = \dfrac{2\,Re(a)}{(z+\bar{a})^2}$

Let $f(a) = |g_a'(\pi)| = \left| \dfrac{2\,Re(a)}{(\pi+\bar{a})^2} \right| =$

## 2 (ii)

① why restrict to real a?

② why restrict to Möbius transf.?

$$g(z) = \frac{z-a}{z+\bar{a}}$$

$$g'(z) = \frac{(z+\bar{a})-(z-a)}{(z+\bar{a})^2} = \frac{\bar{a}+a}{(z+\bar{a})^2}$$

$$= \frac{2\,\mathrm{Re}(a)}{(z+\bar{a})^2}$$

$$f(a) = |g'(\pi)| = \left|\frac{2\,\mathrm{Re}(a)}{(\pi+\bar{a})^2}\right|$$

$$= \frac{2a}{(\pi+a)^2}$$

So $g'(\pi)$ max at $a=\pi$

$$f'(a) = \frac{2(\pi+a)^2 - 4a(\pi+a)}{(\pi+a)^{\cancel{4}\,3}}$$

$$= \frac{2\cancel{(\pi+a)^2}}{(\pi+a)^{\cancel{4}\,2}} - \frac{4a\cancel{(\pi+a)}}{(\pi+a)^{\cancel{4}\,3}}$$

$$= \frac{2}{(\pi+a)^2} - \frac{4a}{(\pi+a)^3}$$

$$f'(a) = 0 \implies \frac{2}{\cancel{(\pi+a)^2}} = \frac{4a}{(\pi+a)^{\cancel{3}}} \implies 2(\pi+a) = 4a$$

$$\implies 2\pi + 2a - 4a = 0$$

$$\implies 2\pi - 2a = 0$$

$$\implies \boxed{a=\pi}\,.$$

(2) $G$ family of all analytic mappings from the half space $\{z : Re(z) > 0\}$ into the open unit disc $\{z : |z| < 1\}$.

Define $\Theta = \sup\{|f'(\pi)| : f \in G\}$.

(i) Find upper bd for $\Theta$.
(ii) Show $\exists\, g \in G$ s.t. $|g'(\pi)| = \Theta$.

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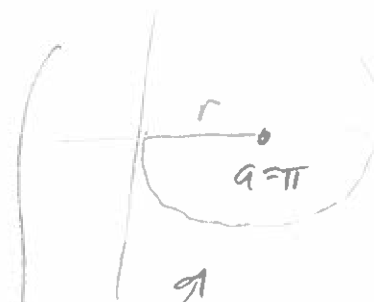
cauchy estimate

(i) Recall $|f^{(n)}(a)| \leq \dfrac{M(r)\, n!}{r^n}$ where $M(r) = \max\limits_{|z-a|=r} |f(z)|$

since $f(z) \in \{w : |w| < 1\}$

$$|f'(\pi)| \leq \frac{M(r) \cdot 1}{r^1} = \frac{1 \cdot 1}{r} = \frac{1}{\pi}$$




r max $= \pi$ since $f$ only defined in right half-plane

So

$$\boxed{\Theta = \sup\{|f'(\pi)| : f \in G\} \leq \frac{1}{\pi}}$$

③ $f$ analytic in $\{0<|z|<\rho\}$.

(i) Define the three possible isolated singularities of $f$ at 0

(ii) Prove $e^f$ cannot have a pole at 0

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(i) Removable singularity: at 0: $|f(z)|$ bdd as $z \to 0$, here $\lim_{z\to 0} z f(z) = 0$, and $f$ can be extended to an analytic function in $|z|<\rho$.

Pole: at 0: $|f(z)| \to \infty$ as $z \to 0$, and $\lim_{z\to 0} z^k f(z)$ is defined; Namely $\exists$ holomorphic $g$ s.t. $f(z) = \dfrac{g(z)}{z^k}$ (where $k$ = order of pole)

Essential singularity at 0: if neither pole nor remov.

(ii) If $f$ has a removable singularity or essential singularity at 0, then clearly $e^f$ will also have a removable or essential, respectively.

Suppose $f$ has a pole at 0. Then $f(z) = \dfrac{p(z)}{z^m} + h(z)$ where $h$ holomorphic.

Then $e^{f(z)} = e^{\frac{p(z)}{z^m}} e^{h(z)}$; $e^{h(z)}$ clearly still holom.

and note that $e^{\frac{p(z)}{z^m}} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{p(z)}{z^m}\right)^n$, hence $e^{\frac{p(z)}{z^m}}$ has an essential singularity since the tail of the Laurent exp. is infinite

So $e^f$ has essential singularity, not pole.

④ $B(r) = \{|z|<r\}$, $f: B(1) \to B(1)$ holomorphic

(i) If $f(0) \in (-1,0)$, show $\frac{1}{2} \notin f(B(\frac{1}{2}))$

Let $f(0)=a$, hence $-1<a<0$, and consider the transformation

$$T(z) = \frac{a-z}{1-\bar{a}z}.$$

Then $T(0)=a$ and $T(a)=0$, and $|T(z)|<1$ for $|z|<1$.

So now see that $T \circ f(0) = T(a) = 0$ and for $|z| \le 1$,

$|T \circ f(z)| \le 1$ since $|f(z)| \le 1$ implies $|T \circ f(z)| < 1$,

hence we may apply Schwarz lemma, i.e.

$$|T \circ f(z)| \le |z|.$$

Now notice that $|T(\frac{1}{2})| = \left|\frac{a-\frac{1}{2}}{1-\bar{a}\frac{1}{2}}\right| \overset{a\in\mathbb{R}}{=} \left|\frac{a-\frac{1}{2}}{1-a\frac{1}{2}}\right|$

and since $-1<a<0$, we have:

$$|a-\tfrac{1}{2}| > \tfrac{1}{2}$$

and since $0<-a<1$, we have $0<-\frac{1}{2}a<\frac{1}{2}$

$$|1-\tfrac{1}{2}a| < \tfrac{3}{2} \Rightarrow \frac{1}{|1-\frac{1}{2}a|} > \frac{2}{3}$$

So we have $|T(\frac{1}{2})| > \frac{\frac{1}{2}}{\frac{2}{3}} = \frac{3}{4}$

Now let $z \in B(\frac{1}{2})$, hence $|z|<\frac{1}{2}$, and suppose $f(z)=\frac{1}{2}$,

i.e. $|T \circ f(z)| = |T(\frac{1}{2})| > \frac{3}{4}$,

but we showed $|T \circ f(z)| \le |z| < \frac{1}{2}$, contradiction.

hence $\nexists z \in B(\frac{1}{2})$ s.t. $f(z)=\frac{1}{2}$,

i.e. $\frac{1}{2} \notin f(B(\frac{1}{2}))$

(ii) If $B(\frac{1}{2}) \subseteq f(B(\frac{1}{2}))$, show $\exists \theta \in \mathbb{R}$ s.t. $f(x) = e^{i\theta} z \quad \forall z \in B(1)$