check #3

Complex analysis, Graduate Exam Fall 2006

Answer all four questions. Partial credit will be given to partial solutions.

1. Let a be a real number with $a > e$. Show that the equation $e^z = az^n$ has n solutions inside the unit circle.

2. Find the Fourier transform $\hat{f}(\omega)$ of the function $f(x) = \frac{1}{1+x^2}$, $x \in \mathbb{R}$,

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} \frac{e^{-i\omega x}}{1+x^2}\, dx.$$

3. Fix $\tau \in \mathbb{C}$ with non-zero imaginary part. Let $f(z)$, $z \in \mathbb{C}$ be a non-constant meromorphic function such that $f(z) = f(z + m + n\tau)$ for all $m, n \in \mathbb{Z}$ (such functions are called doubly periodic). Show that f has infinitely many singularities.

✓ **4.** Let $f(z)$ be a one to one conformal map from the unit disk to a square with center 0 such that $f(0) = 0$. Show that $f(iz) = if(z)$. Show also that if $f(z)$ is given by the power series $f(z) = \sum_{n=1}^{\infty} c_n z^n$ then $c_n = 0$ for all n such that $n - 1$ is not divisible by 4.

$|f(z) - g(z)| < |f(z)|$

$p(z) = 2z^5 + 4z^2 + 1$

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① $a > e$; show that $e^z = az^n$ has n sol'ns inside unit circle

Let $g(z) = az^n - e^z$, $f(z) = az^n$ and apply Rouché's theorem on the unit circle;

$$|f(z) - g(z)| = |e^z| = e^{\operatorname{Re} z} \leq e^1 = e < a = a|z|^n$$

since z is on the unit circle

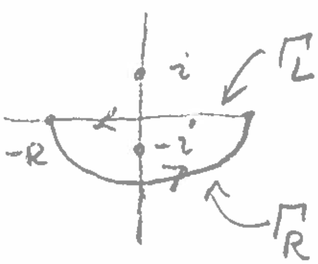
$$= |az^n|$$
$$= |f(z)|.$$

So by Rouché, g has the same # of roots as f, and $f(z) = az^n$ has n roots in the unit circle (0 w/ mult. n), hence $e^z = az^n$ has n sol'ns.

② Compute $\int_{-\infty}^{\infty} \frac{e^{-i\omega x}}{1+x^2}\,dx$

so $= \pi e^{-\omega}$ since contour is ccw

Consider the contour integral $\int_\Gamma \frac{e^{-i\omega z}}{1+z^2}\,dz$ where Γ is:



Then $\int_\Gamma \frac{e^{-i\omega z}}{1+z^2}\,dz = 2\pi i \operatorname*{Res}_{z=-i} \frac{e^{-i\omega z}}{1+z^2}$

$$= 2\pi i \lim_{z \to -i} \frac{e^{-i\omega z}\cancel{(z+i)}}{\cancel{(z+i)}(z-i)}$$

Now, $\int_\Gamma = \int_{\Gamma_L} + \int_{\Gamma_R}$, so WTS $\int_{\Gamma_R} \to 0$ as $R \to \infty$

$$= 2\pi i \left(\frac{e^{-\omega}}{-2i}\right) = -\pi e^{-\omega}$$

$$\left|\int_{\Gamma_C} \frac{e^{-i\omega z}}{1+z^2}\,dz\right| \leq \int_{\Gamma_C} \frac{|e^{-i\omega z}|}{|1+z^2|}\,dz$$

and $|e^{-i\omega z}| = e^{\omega y} < 1$ on the lower halfplane since $y \leq 0$ here.
(since $\operatorname{Re}(-i\omega z) = \operatorname{Re}(-i\omega x + \omega y) = \omega y$)

Now see that: $\int_{\Gamma_C} \frac{|e^{i\omega z}|}{|1+z^2|}|dz| \leq \int_{\Gamma_C} \frac{|dz|}{|1+z^2|} = \int_{\pi}^{2\pi} \frac{|Rie^{i\theta}d\theta|}{|1+R^2e^{2i\theta}|} = \int_{\pi}^{2\pi} \frac{R\,d\theta}{|1+R^2e^{2i\theta}|}$

and clearly $|1+R^2e^{2i\theta}|$ ranges from $|1-R^2|$ to $|1+R^2|$ (min to max real part), so $|1+R^2e^{2i\theta}| \geq R^2-1$, and

$$\leq \frac{R}{R^2-1}\int_{\pi}^{2\pi} d\theta = \frac{\pi R}{R^2-1} \to 0 \text{ as } R \to \infty.$$

③ Fix $\tau \in \mathbb{C}$ w/ $\text{Im}(\tau) \neq 0$. Let $f(z)$ be non const. meromorphic fn. such that $f(z) = f(z+m+n\tau)$ $\forall m,n \in \mathbb{Z}$. Show f has infinitely many singularities

Suppose that f has no singularities, hence f is entire.
Now consider the region $B_R = \{|z| \leq R\}$ such that $0, 1+\tau \in B_R$
f is continuous, hence bounded on this region, and so for all $z \in \mathbb{C} \setminus B_R$,
$f(z) = f(z+m+n\tau)$ for some $m,n \in \mathbb{Z}$ with $z+m+n\tau \in B_R$, hence
f is bounded everywhere, and therefore constant by Liouville,
a contradiction.

Now, f must have a singularity. Since it is doubly periodic,
it must repeat itself infinitely many times, hence infinite
# of singularities.

④ f one-to-one conformal map from $D = \{|z|<1\}$ to a square with center 0 such that $f(0)=0$.

Show (a) $f(iz) = if(z)$ and (b) for $f(z) = \sum_{n=1}^{\infty} c_n z^n$, $c_n = 0$ for $n \not\equiv 1 \pmod 4$

(a) Note that $f(iz) = if(z) \iff -if^{-1}(if(z)) = z$

So define the map $g(z) = -if^{-1}(if(z))$; see first that:

$$\begin{tikzcd} D \arrow[r, "f"] \arrow[rrrr, bend left, "g"] & S \arrow[r, "i"] & S \arrow[r, "f^{-1}"] & D \arrow[r, "-i"] & D \end{tikzcd}$$

rotating the square by $\pi/2$ yields same square

, hence $g: D \to D$ analytic ①

And $g(0) = -if^{-1}(if(0)) = -if^{-1}(0) = -i(0) = 0 \Rightarrow g(0)=0$ ②

Finally, consider $|g'(0)|$:

$$|g'(0)| = \left|\frac{d}{dz}\left(-if^{-1}(if(z))\right)\Big|_{z=0}\right| = \left|-i\frac{d}{dz}\left(f^{-1}(if(z))\right)\Big|_{z=0}\right|$$

$$= \left|-i(f^{-1})'(if(0)) \cdot \frac{d}{dz}(if(z))\Big|_{z=0}\right| = \left|-i(f^{-1})'(0)\, i f'(0)\right|$$

$$= \left|(f^{-1})'(0) f'(0)\right|$$

But now see that $f^{-1} \circ f(z) = z \Rightarrow (f^{-1} \circ f)'(z) = 1$

$\Rightarrow (f^{-1})'(f(0)) f'(0) = 1 \Rightarrow (f^{-1})'(0) \cdot f'(0) = 1$

$\Rightarrow |g'(0)| = 1$ ③

Hence ① + ② + ③ $\Rightarrow g(z) = cz$ for some $|c| = 1$,

$\Rightarrow -if^{-1}(if(z)) = cz \Rightarrow if(z) = f(icz)$.

But now $if'(z) = f'(icz)(ic)$, and let $z=0$,

here $if'(0) = f'(0)ic \Rightarrow f'(0) = cf'(0) \Rightarrow c = 1$ since $f'(0) \neq 0$ by conformality of f.

hence $if(z) = f(iz)$

(b) Consider the power series $f(z) = \sum_{n=1}^{\infty} c_n z^n$.
and recall $f(iz) = if(z)$, hence:

$$\sum_{n=1}^{\infty} c_n (iz)^n = i \sum_{n=1}^{\infty} c_n z^n$$

$$\Rightarrow \sum_{n=1}^{\infty} c_n (i)^n z^n = \sum_{n=1}^{\infty} c_n i z^n. \Rightarrow \sum_{n=1}^{\infty} c_n (i)^{n-1} i z^n = \sum_{n=1}^{\infty} c_n i z^n$$

So now we want n such that $(i)^{n-1} = i$; in fact, since $i^4 = i$, this is only true if $4 \mid n-1 \Leftrightarrow n \equiv 1 \pmod 4$
Hence for the power series to be equal, $c_n \underline{= 0 \text{ otherwise}}$