COMPLEX ANALYSIS GRADUATE EXAM

FALL 2005

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper. D denotes the open unit disc $\{z \in \mathbf{C} : |z| < 1\}$ in the complex plane.

Half-Residue

1. Show that the integral

$$\int_{-\infty}^{\infty} \frac{\cos x}{4x^2 - \pi^2}\, dx$$

exists, and evaluate it.

(i)-Max Mod

(ii)-winding # + continuity

2. Suppose that f is analytic on the open unit disc D and continuous on $\overline{D}$, and is non-constant on D. Suppose also there exist constants $0 < a \le b$ so that $a \le |f(z)| \le b$ for $|z| = 1$.
   (i) Show that $f(D) \subset B(0, b)$.
   (ii) Show that either $f(D) \supset B(0, a)$ or $f(D) \cap B(0, a) = \emptyset$.

Liouville

3. Suppose that entire functions f and g satisfy $|f(z)| \le |g(z)|$ for all $z \in \mathbf{C}$. Show that there exists $\lambda \in \mathbf{C}$ such that $f = \lambda g$.

conformal

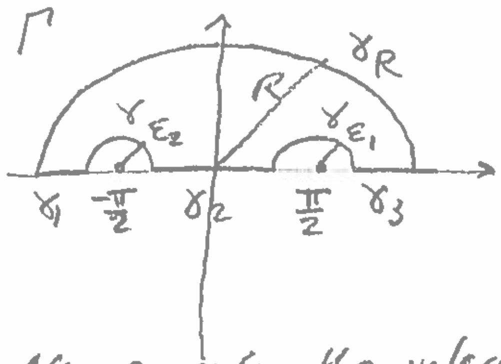
4. Let Ω be the region between the circles $C_1 : |z| = 1$ and $C_2 : |z - 1/2| = 1/2$. Find a conformal mapping of Ω onto the open unit disc. [You may give your answer as the composition of several mappings, so long as each mapping is precisely described.]

CA - Fall 05

(1.) $\int_{-\infty}^{\infty} \frac{\cos x}{4x^2-\pi^2}\,dx$

Consider the function $f(z) = \frac{e^{iz}}{4z^2-\pi^2}$ with poles $z = \pm\frac{\pi}{2}$

Now consider $\int_\Gamma f(z)\,dz$ where Γ is the contour:



see that, as $\varepsilon_1, \varepsilon_2 \to 0$, $R \to \infty$ the integral

$$\int_{-\infty}^{\infty} f(z)\,dz = \int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3}$$

• Now consider the integral $\int_{\gamma_R} f(z)\,dz$:

since $\theta \in (0,\pi)$ hence $\sin\theta$ positive

$$\left|\int_{\gamma_R} f(z)\,dz\right| \le \int_{\gamma_R} \frac{|e^{iz}|}{|4z^2-\pi^2|}|dz| \le \int_{\gamma_R} \frac{e^{-R\sin\theta}}{4R^2-\pi^2}|dz| \le \frac{1\cdot \pi R}{4R^2-\pi^2} \to 0$$
as $R \to \infty$

So as $R \to \infty$,

$$\int_\Gamma = \int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} + \int_{\gamma_{\varepsilon_1}} + \int_{\gamma_{\varepsilon_2}}$$

However, note that Γ contains no poles, hence $\int_\Gamma = 0$,

so we have $\int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} = -\left(\int_{\gamma_{\varepsilon_1}} + \int_{\gamma_{\varepsilon_2}}\right)$

• Now consider the integrals $\int_{\gamma_{\varepsilon_1}}, \int_{\gamma_{\varepsilon_2}}$:

Since $f(z)$ has simple poles at $\frac{\pi}{2}, -\frac{\pi}{2}$, we may rewrite f as follows:

$$f(z) = \frac{a}{z-\frac{\pi}{2}} + h_1(z), \quad f(z) = \frac{b}{z+\frac{\pi}{2}} + h_2(z)$$

where h_i are holomorphic at $\frac{\pi}{2}, -\frac{\pi}{2}$ for $i=1,2$ resp.

Note that, by def'n, $a = \operatorname{Res}_{z=\frac{\pi}{2}} f$, $b = \operatorname{Res}_{z=-\frac{\pi}{2}} f$

Now consider $\int_{\gamma_{\varepsilon_1}} f(z)dz$ and $\int_{\gamma_{\varepsilon_2}} f(z)dz$:

$$\int_{\gamma_{\varepsilon_1}} f(z)dz = \int_{\gamma_{\varepsilon_1}}\left(\frac{a}{z-\frac{\pi}{2}} + h_1(z)\right)dz = \int_{\gamma_{\varepsilon_1}} \frac{a}{z-\frac{\pi}{2}}dz + \int_{\gamma_{\varepsilon_1}} h_1(z)\,dz$$

Now, since h_1 holomorphic, h_1 is bdd near $\frac{\pi}{2}$, and as $\varepsilon_1 \to 0$, length$(\gamma_{\varepsilon_1}) \to 0$, hence $\int_{\gamma_{\varepsilon_1}} h_1(z)dz \to 0$ as $\varepsilon_1 \to 0$.

Therefore $\int_{\gamma_{\varepsilon_1}} f(z)dz \longrightarrow \int_{\gamma_{\varepsilon_1}} \frac{a}{z-\frac{\pi}{2}}dz = \pi i a$ as $\varepsilon_1 \to 0$.

Similarly, $\int_{\gamma_{\varepsilon_2}} f(z)dz \longrightarrow \pi i b$ as $\varepsilon_2 \to 0$.

So then, as $\varepsilon_i \to 0$ and $R \to \infty$,

$$\int_{-\infty}^{\infty} \frac{e^{ix}\,dx}{4x^2-\pi^2} = \int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} = -(\pi i a + \pi i b)$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{e^{ix}\,dx}{4x^2-\pi^2} = -\pi i(a+b)$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\cos x + i\sin x}{4x^2-\pi^2}dx = -\pi i(a+b)$$

$$\Rightarrow \underline{\int_{-\infty}^{\infty} \frac{\cos x}{4x^2-\pi^2}dx} = \mathrm{Re}(-\pi i(a+b))$$

Now $a = \operatorname{Res}_{z=\frac{\pi}{2}} \frac{e^{iz}}{4z^2-\pi^2} = \left.\frac{e^{iz}\cancel{(z-\frac{\pi}{2})}}{4\cancel{(z-\frac{\pi}{2})}(z+\frac{\pi}{2})}\right|_{z=\frac{\pi}{2}} = \frac{e^{\frac{i\pi}{2}}}{4(\frac{\pi}{2}+\frac{\pi}{2})} = \frac{i}{4\pi}$

$b = \operatorname{Res}_{z=-\frac{\pi}{2}} \frac{e^{iz}}{4z^2-\pi^2} = \frac{e^{-i\frac{\pi}{2}}}{4(-\frac{\pi}{2}-\frac{\pi}{2})} = \frac{-i}{-4\pi} = \frac{i}{4\pi}$

$$\Rightarrow -\pi i\left(\frac{i}{4\pi} + \frac{i}{4\pi}\right) = -\cancel{\pi} i\left(\frac{2i}{4\cancel{\pi}}\right) = \frac{1}{2}$$

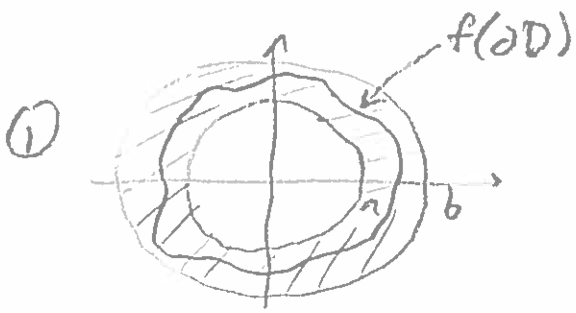
(2.) f analytic on $D=\{|z|<1\}$, continuous on $\overline{D}$, non-const. on D.
Suppose also that $\exists\ 0<a\le b$ s.t. $a\le |f(z)|\le b$ for $|z|=1$

(i) Show $f(D)\subseteq B_b(0)$:

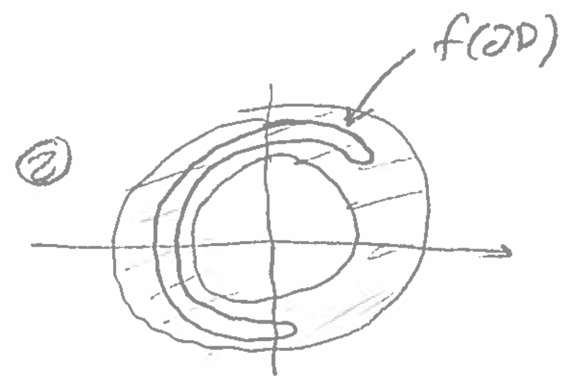
Since f is cont. on $\overline{D}$, by the maximum modulus principle it must attain its maximum on $\partial\overline{D}$.
Now, by hypothesis, for $z\in\partial\overline{D}$, $|f(z)|\le b$, but by maximum modulus no $z\in D$ can have $|f(z)|$ greater than the max on $\partial\overline{D}$, hence $|f(z)|\le b$ for all $z\in D$, hence $f(D)\subseteq B_b(0)$.

(ii) Show $B_a(0)\subseteq f(D)$ or $f(D)\cap B_a(0)=\phi$.

Note that restricting the function f to ∂D defines a closed curve in $\mathbb{C}$; it will be contained in the annulus $a\le |z|\le b$ and it will either have 0 or non-zero winding number about 0:



Non-zero winding number for 0



zero winding # for 0

Now, since f is continuous on $\overline{D}$, we have $f(\overline{D})$ compact, and by the open mapping theorem, $f(D)$ is open. (since f analytic on D). We know that $f(D)\subseteq f(\overline{D})$, and since $f(\overline{D})$ cpt, so must $\overline{f(D)}\subseteq f(\overline{D})$, hence the interior of the curve ∂D is mapped to the interior of $f(\partial D)$, hence case(1) $\Rightarrow f(D)\supseteq B_a(0)$

case(2) $\Rightarrow f(D)\cap B_a(0)=\phi$.

(3.) f, g entire such that $|f(z)| \le |g(z)|$ for all z.
Show $f = \lambda g$:

Consider the function $h(z) = \dfrac{f(z)}{g(z)}$.

Since f and g are entire, h will only have isolated singularities at the zeroes of g.
Since g is entire these singularities will only be poles or removable.
But $|f(z)| \le |g(z)| \Rightarrow |h(z)| = \left|\dfrac{f(z)}{g(z)}\right| \le 1$, hence h is bounded everywhere, and in particular is bdd near the singularities, hence they must be removable.

Therefore we may take h to be entire; we just showed that h is bdd, hence $h = \text{const.}$ by Liouville.

Then $\dfrac{f(z)}{g(z)} = \lambda$ for some $\lambda \in \mathbb{C}$ $\Rightarrow$ $f(z) = \lambda g(z)$

(4.) Conformally map $\Omega = \{\text{area b/t } \{|z|=1\} \text{ and } \{|z-\tfrac{1}{2}| = \tfrac{1}{2}\}\}$ onto the unit disc :