COMPLEX ANALYSIS GRADUATE EXAM

Fall **Spring 2015**

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning, and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Evaluate the integral

$$\int_0^\infty \frac{\sin^2 x}{x^2}\, dx,$$

being careful to justify your answer.

2. Determine the number of roots of $f(z) = z^9 + z^6 + z^5 + 8z^3 + 1$ inside the annulus $1 < |z| < 2$.

3. Suppose that f is holomorphic on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and suppose that for $z \in \mathbb{D}$ one has $\Re(f(z)) > 0$ and $f(0) = 1$. Prove that $|f(z)| \leq \frac{1+|z|}{1-|z|}$ for all $z \in \mathbb{D}$.

4. For $a_n = 1 - \frac{1}{n^2}$, let

$$f(z) = \prod_{n=1}^{\infty} \frac{a_n - z}{1 - a_n z}.$$

(1) Show that f defines a holomorphic function on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.
(2) Prove that f does not have an analytic continuation to any larger disk $\{z \in \mathbb{C} : |z| < r\}$ for some $r > 1$.

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(1) Compute $\displaystyle\int_0^\infty \frac{\sin^2 X}{X^2}\,dX.$

The integral exists. so we may apply IBP:

$$\int_0^\infty \frac{\sin^2 X}{X^2}\,dX = \underbrace{-\frac{\sin^2 X}{X}\Big|_0^\infty}_{\lim_{X\to 0}\frac{\sin X}{X}\sin X - \lim_{X\to\infty}\frac{\sin^2 X}{X} \,=\, 0 - 0.} + 2\int_0^\infty \frac{\sin X\cos X}{X}\,dX$$

$$= \int_0^\infty \frac{\sin(2X)}{X}\,dX = \int_0^\infty \frac{\sin X}{X}\,dX.$$

$u = 2X$
$du = 2dX$

$$2\int_0^\infty \frac{\sin X}{X} \overset{Im}{=} \left(p.v.\int_{-\infty}^{\infty} \frac{e^{iX}}{X}\right) \overset{Im}{=} (\pi i \cdot 1) = \pi$$

$$\Rightarrow \int_0^\infty \frac{\sin^2 X\,dX}{X^2} = \int_0^\infty \frac{\sin X\,dX}{X} = \pi/2. \quad \square$$