COMPLEX ANALYSIS GRADUATE EXAM

Spring 2014

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. For $a > 0$, evaluate the integral

$$\int_0^\infty \frac{\log x}{(a+x)^3}\, dx,$$

being careful to justify your methods.

2. Find a conformal mapping of the region $\{z : |z| > 1\} \setminus (1, \infty)$ onto the open unit disc $\{z : |z| < 1\}$. You may give your answer as the composition of several mappings, so long as each mapping is precisely described.

3. Suppose that f_n are analytic functions on a connected open set $U \subset \mathbb{C}$ and that $f_n \to f$ uniformly on compact subsets of U. In each case indicate the main steps in the proofs of the following standard results.

(i) f is analytic in U;

(ii) $f_n' \to f'$ uniformly on compact subsets of U;

(iii) if $f_n(z) \neq 0$ for all n and all $z \in U$, then either $f(z) \neq 0$ for all $z \in U$ or else $f \equiv 0$.

4. (a) Suppose that f is analytic on the open unit disc $\{z : |z| < 1\}$ and that there exists a constant M such that $|f^k(0)| \leq k^4 M^k$ for all $k \geq 0$. Show that f can be extended to be analytic on $\mathbb{C}$.

(b) Suppose that f is analytic on the open unit disc $\{z : |z| < 1\}$ and that there exists a constant $M > 1$ such that $|f(1/k)| \leq M^{-k}$ for all $k \geq 1$. Show that f is identically zero.

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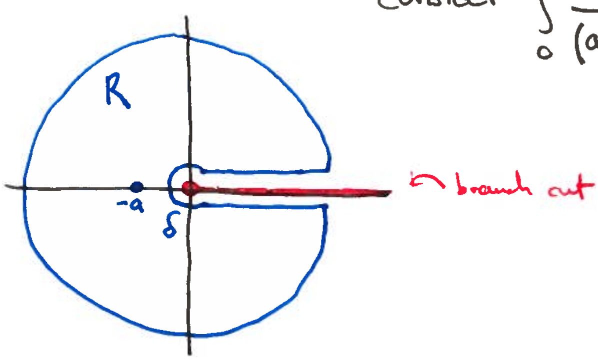
(1) For $a>0$, compute $\displaystyle\int_0^\infty \frac{\log x}{(a+x)^3}\,dx$.

Consider $\displaystyle\int_0^\infty \frac{\log^2 x}{(a+x)^3}\,dx$.



The integrals along the arcs vanish:

$$\int_0^{2\pi} \frac{R\log^2 R}{(R-|a|)^3}\,dt \to 0 \text{ as } R\to\infty$$

since $R \gg \log^2 R$ as $R\to\infty$.

$$\int_0^{2\pi} \frac{\delta\log^2 \delta}{(|a|-\delta)^3} \to 0 \text{ as } \delta\to 0$$

since $\delta\log^2\delta \to 0$ as $\delta\to 0$.

The integral becomes $\displaystyle\int_0^\infty \frac{\log^2 x-(\log x+2\pi i)^2\,dx}{(a+x)^3}$

$$=\int_0^\infty \frac{4\pi^2-4\pi i\log x}{(a+x)^3}\,dx = 2\pi i\,\mathrm{Res}(-a) = \frac{2\pi i}{2}\frac{d^2}{dz^2}\log^2 z\Big|_{z=-a}$$

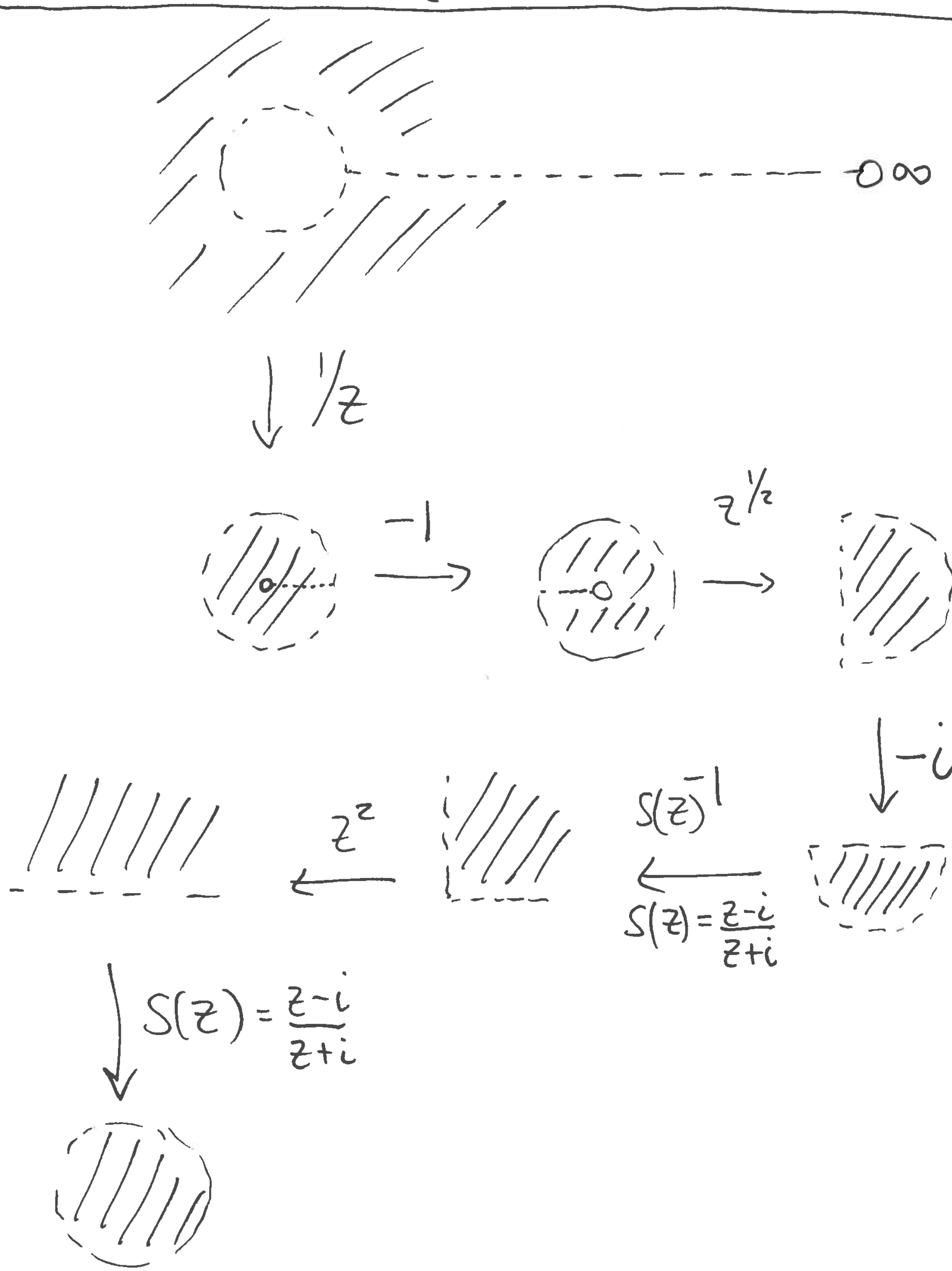
$\frac{2\log z}{z}$

$2\left(\frac{1}{z^2}-\frac{\log z}{z^2}\right)$

$$=\frac{2\pi i}{a^2}\left(1-\pi i-\log(a)\right) \overset{\text{imag}}{\Longrightarrow} (-4\pi)\int_0^\infty \frac{\log x}{(a+x)^3}\,dx = \frac{2\pi}{a^2}(1-\log a)$$

$\Longrightarrow$ integral $= (\log a - 1)/2a^2$. □

(2) Conformally map $\{z : |z| > 1\}$ onto the open unit disk.



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