COMPLEX ANALYSIS GRADUATE EXAM

Fall 2013

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning, and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Compute

$$\int_0^\infty \frac{\log^2 x}{1+x^2}\,dx.$$

2. Find the number of *distinct* zeros of $f(z) = z^6 + (10-i)z^4 + 1$ inside $(-1,1)\times(-1,1)$.

3. Suppose that f is holomorphic in a neighborhood U of $a \in \mathbb{C}$. Consider the following two statements:

(i) There exist two sequences $\{z_k\}_{k=1}^\infty$ and $\{w_k\}_{k=1}^\infty$ in $U\setminus\{a\}$ converging to a such that $z_k \neq w_k$ and $f(z_k) = f(w_k)$ for all $k \in \mathbb{N}$.

(ii) $f'(a) = 0$.

Determine whether either of the statements implies the other one. In each case justify your answer with a proof or a counterexample.

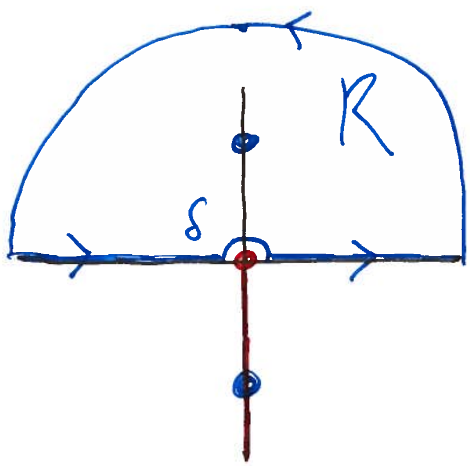
4. Let f be analytic in an open set $U \subseteq \mathbb{C}$, and let $K \subseteq U$ be compact. Show that there exists a constant C depending on U and K such that

$$|f(z)| \leq C\left(\int_U |f|^2\right)^{1/2}$$

for all $z \in K$.

Complex, Fall 2013

(1) Compute $\displaystyle\int_0^\infty \frac{\log^2 x}{1+x^2}\,dx$



Branch Cut

$\log re^{ix} = \ln r + ix$

$r>0,\ -\frac{\pi}{2} < x < \frac{3\pi}{2}$

$$2\pi i \,\mathrm{Res}(i) = 2\pi i \cdot \frac{\log^2(i)}{2i}$$

$$= \pi\left(\frac{\pi i}{2}\right)^2 = -\frac{\pi^3}{4}.$$

$|\cdot|$ on large semicircle :

$$\leq \int_0^\pi \frac{R\,|\log R|^2 + Rt^2}{R^2-1}\,dt$$

$\longrightarrow 0$ as $R\to\infty$ since

$$\lim_{R\to\infty}\frac{\log^2 R}{R} = 0 \text{ and } \lim_{R\to\infty}\frac{1}{R} = 0.$$

$|$On small semicircle :

$$\leq \int_0^\pi \frac{\delta\,\log^2\delta + \delta t^2}{1-\delta^2}\,dt \longrightarrow 0 \text{ as } \delta\to 0$$

since $\displaystyle\lim_{\delta\to 0}\delta\log^2\delta = 0.$

And calculating on the negative x-axis in reverse, i.e. $\gamma(t) = -t$, $\log^2(-t)$

$$= (\ln t + \pi i)^2 = \log^2 t - \pi^2 + 2\ln t\,\pi i \text{, gives}$$

$$+ \int_0^\infty \frac{\log^2 t - \pi^2 + 2\ln t\,\pi i}{1+t^2}\,dt.$$

$$-\int_0^\infty \frac{\pi^2}{1+t^2}\,dt = -\frac{\pi^3}{2}.$$

Taking real parts in the Residue Theorem

$$2\int_0^\infty \frac{\log^2 t\,dt}{1+t^2} = \frac{\pi^3}{2} - \frac{\pi^3}{4} = \frac{\pi^3}{4}$$

$$\therefore \int_0^\infty \frac{\log^2 t\,dt}{1+t^2} = \pi^3/8. \qquad \square$$

(2) Find the number of distinct zeroes of $f(z) = z^6 + (10-i)z^4 + 1$ inside $[-1,1] \times [-1,1]$.

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claim: # zeroes of $(10-i)z^4 + 1$ inside $\{z : |z| < 1\}$ is 4 (including multiplicity).

pf(claim): $|z^4|^2 = \left|\frac{-1}{10-i}\right|^2 = \frac{1}{101} \Rightarrow |z| < 1$. ✓

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claim: $|z^6| < |z^6 + (10-i)z^4 + 1|$

on $|z| = 1$. $\therefore$ $f(z)$ has 4 zeroes in $|z|<1$.

pf(claim): $|z^6 + (10-i)z^4 + 1|$

$\geq 10 - 1 - 1 > 1$. The conclusion follows by Rouché! ✓

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claim : $|z^6| < |z^6 + (10-i)z^4 + 1|$

on $\{|z|=2\}$. $\therefore f(z)$ has 4 zeroes

inside $\{|z|<2\}$. $\therefore f(z)$ has 4 zeroes

inside $[-1,1] \times [-1,1]$.

pf (claim) : $|z^6 + (10-i)z^4 + 1|$

$\geq 160 - 64 - 1 = 95 > 64.$ ✓

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claim: The four zeroes of $f(z)$ inside $[-1,1] \times [-1,1]$ are distinct.

pf (claim) : If this were not true, then $f(z)$ and $f'(z)$ would have a zero in common.

$$f'(z) = 6z^5 + (40-4i)z^3$$
$$= 2z^3\left(3z^2 + (20-2i)\right).$$

Since $z=0$ is not a root of $f(z)$, then there is a root z of $f(z)$ satisfying

$$z^2 = \frac{2i-20}{3}.$$

Consider
$$\left|\left(\frac{2i-20}{3}\right)^3 + (10-i)\left(\frac{2i-20}{3}\right) + 1\right| = 0. \quad (*)$$

norm^2 of $(10-i)\left(\frac{2i-20}{3}\right)$ is $101 \cdot \left(\frac{4}{9} + \frac{400}{9}\right)$
$\le 11^2 \cdot \frac{21^2}{3^2}$

$$\Rightarrow \left|(10-i)\left(\frac{2i-20}{3}\right)\right| \le 11 \cdot \frac{21}{3}.$$

Since $\left(\frac{4}{9} + \frac{400}{9}\right) \geq \frac{20^2}{3^2}$, $\left|\left(\frac{2i-20}{3}\right)^3 + 1\right| \geq \frac{20^3}{3^3} + 1$.

so $(*) \geq$

$$\frac{20^3}{3^3} + 1 - 11 \cdot \frac{21}{3} \geq \frac{20}{3} \cdot \frac{20^2}{3} \cdot \frac{1}{3} - 3 \cdot \frac{11}{3} \cdot \frac{21}{3} > 0.$$

$\left(\begin{array}{r} 33 \\ \times 21 \\ \hline 33 \\ 660 \\ \hline 693 < 8000 \end{array}\right)$

$0 \neq$. $\therefore$ 4 distinct roots. $\square$

(3) Suppose f is holomorphic in $U \ni a$ open.

Consider the following statements:

(i) There exist $\{z_k\}$ and $\{w_k\}$ in $U - \{a\}$ converging to a s.t. $z_k \neq w_k$ and $f(z_k) = f(w_k)$ $\forall k$.

(ii) $f'(a) = 0$.

Consider (i)$\Rightarrow$(ii) & (ii)$\Rightarrow$(i). T/F?

(i) $\Longrightarrow$ (ii) is true, since not (ii) $\Rightarrow$ not (i) is true.

Indeed, if $f'(a) \neq 0$, then by the inverse function theorem for analytic functions $f(z)$ is bijective locally around a, so two such sequences $\{z_k\}$ & $\{w_k\}$ cannot exist.

(ii) $\Longrightarrow$ (i) is also true. Assume $f'(a) = 0$ and $f(z) \neq$ constant. Obviously we may assume $f(a) = 0$. Then $f(z) = (z-a)^k g(z)$ where $g(a) \neq 0$, $k \geq 2$. Then on a small disk around a, by continuity we may choose a branch of $z^{1/k}$ analytic on the image of g on this disk, i.e. we may write $f(z) = [(z-a) g(z)^{1/k}]^k$. (*)

(*) By the open mapping theorem, $(z-a)g(z)^{1/k}$ maps onto a disk about 0. So $f(z)$ is at least k-to-1 locally.

~~the fibers of $(z-a)^k$ on this small disk contain $k \geq 2$ elements, we see that $f(z)$ is, in fact, locally k-to-1, hence such a sequence $\{z_n\}$ & $\{w_n\}$ exist, upon taking arbitrarily small disks around a~~ $\square$

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(4) f analytic on U, $K \subset U$ compact. Show there is a constant C depending on U and K s.t. $|f(z)| \leq C\left(\int_U |f|^2\right)^{1/2}$ for all $z \in K$.

By the Lebesgue number lemma, choose an $r>0$ s.t. $B(z,r) \subset U$ for all $z \in K$.

$$|f(z)| = \frac{1}{\pi r^2} \iint_{B(z,r)} |f(re^{i\theta})| \leq \frac{1}{\pi r^2} \int_U |f| \overset{\text{Hölder}}{\leq} \frac{1}{\pi r^2} \left[\int_U |f|^2\right]^{1/2}.$$

$\square$