COMPLEX ANALYSIS GRADUATE EXAM

Fall 2012

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Evaluate the integral
$$\int_0^\infty \frac{dx}{1+x^n}, n \geq 2,$$
being careful to justify your methods.

2. Find the Laurent series expansion for
$$\frac{1}{z(z+1)}$$
valid in $\{1 < |z-1| < 2\}$.

3. Suppose that f is an entire function and that there is a bounded sequence of distinct real numbers $a_1, a_2, a_3, \ldots$ such that $f(a_k)$ is real for each k. Show that $f(x)$ is real for all real x.

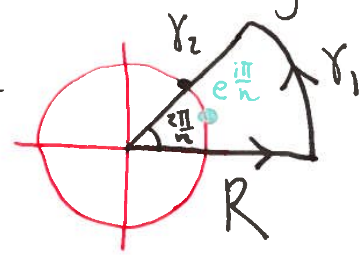
4. Suppose
$$f_n(z) = \sum_{k=0}^{n} \frac{1}{k!z^k}, z \neq 0$$
and let $\varepsilon > 0$. Show that for large enough n, all the zeros of f_n are in the disk $D(0, \varepsilon)$ with center 0 and radius ε.

Complex, Fall 2012

(1) Evaluate $\displaystyle\int_0^\infty \frac{dx}{1+x^n}$ $n \geq 2$.

The integral converges.

The roots of $1+x^n$ are $\{e^{\frac{i\pi}{n}}\zeta_n\}$ where ζ_n is an n^{th} root of unity. We integrate $\frac{1}{1+z^n}$ over the contour



in which there is one singularity at $z = e^{\frac{i\pi}{n}}$ and on which there are no singularities.

Along the arc the integral is

$$i\int_0^{2\pi/n} \frac{Re^{it}\,dt}{1+R^n e^{int}} \quad \text{w/ magnitude} \leq \int_0^{2\pi/n} \frac{R}{R^n-1} \longrightarrow 0 \text{ as } R \to \infty.$$

Along the ray $\gamma_2(t) = (R-t)e^{\frac{i2\pi}{n}}$

the integral is

$$-e^{\frac{i2\pi}{n}} \int_0^R \frac{dt}{1+(R-t)^n \underbrace{e^{i2\pi}}_{=1}} = -e^{\frac{i2\pi}{n}} \int_0^R \frac{dt}{1+t^n}.$$

The residue theorem gives, in the limit $R \to \infty$,

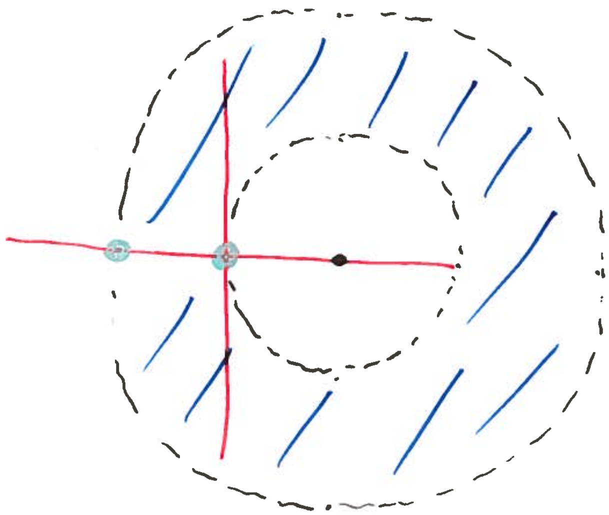
$$\int_0^\infty \frac{dx}{1+x^n}\left(1-e^{\frac{i2\pi}{n}}\right) = 2\pi i \operatorname{Res}\left(e^{\frac{\pi i}{n}}\right)$$

$$\operatorname{Res}\left(e^{\pi i/n}\right) = \frac{1}{n\left(e^{\frac{\pi i}{n}}\right)^{n-1}} \Rightarrow \int_0^\infty \frac{dx}{1+x^n}$$

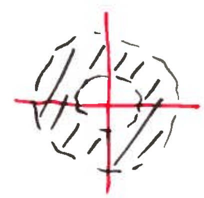
$$= \frac{\pi}{n} \frac{2i}{\left(1-e^{\frac{i2\pi}{n}}\right)\underbrace{e^{\pi i\frac{n-1}{n}}}_{-e^{-\frac{\pi i}{n}}}} = \frac{\pi}{n} \frac{2i}{e^{\frac{i\pi}{n}}-e^{-\frac{\pi i}{n}}}$$

$$= \frac{\pi/n}{\sin(\pi/n)}. \quad \square$$

(2) Find the Laurent expansion of $\frac{1}{z(z+1)}$ in $1 < |z-1| < 2$.



This is the same as finding the expansion for $1<|w|<2$, $w = z-1$, of $\frac{1}{(w+1)(w+2)}$.



We need to find the expansions for $|w|>1$ of $\frac{1}{w+1}$, and for $|w|<2$ of $\frac{1}{w+2}$.

$$|w|>1: \quad \frac{1}{1-(-w)} = \frac{1}{w}\,\frac{1}{1-\left(\frac{-1}{w}\right)} = \frac{1}{w}\sum_{n=0}^{\infty}\left(\frac{-1}{w}\right)^n.$$

$$|w|<2: \quad \frac{1}{2-(-w)} = \frac{1}{2}\,\frac{1}{1-\left(\frac{-w}{2}\right)} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{-w}{2}\right)^n.$$

Now we solve $\frac{1}{(w+1)(w+2)} = \frac{A}{w+1} + \frac{B}{w+2} \Rightarrow 1 = A(w+2)+B(w+1)$

$\Rightarrow A+B=0$ & $1=2A+B$

$\Longrightarrow A = 1$ & $B = -1$.

Replacing $\omega = z - 1$, we conclude

$$\frac{1}{z(z+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(z-1)^{n+1}} + \sum_{n=0}^{\infty} \left(\frac{-1}{2}\right)^{n+1} (z-1)^n$$

for $1 < |z-1| < 2$. $\square$

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(3) f entire. Bounded sequence $\{a_k\}$ of real numbers such that $f(a_k) \in \mathbb{R}$ for all k. Show $f(x) \in \mathbb{R}$ for all $x \in \mathbb{R}$.

It suffices to show f can be expanded in a power series w/ real coefficients; since f is entire, $f(\mathbb{R}) \subset \mathbb{R}$. Since $\{a_k\}$ is bounded, there is a subsequence

$a_{k_\ell} \to a$. Since $f(a_{k_\ell}) \in \mathbb{R}$ $\forall \ell$, by continuity $f(a) \in \mathbb{R}$. We claim $f^{(n)}(a) \in \mathbb{R}$ for all n; it follows that

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \quad , \; a_n = \frac{f^{(n)}(a)}{n!} \; ,$$

can be expanded in a power series w/ real coefficients, as desired.

We've seen already that $f(a) \in \mathbb{R}$.

$$f'(a) = \lim_{z \to a} \frac{f(z) - f(a)}{z - a} \; ;$$

taking this limit along the sequence $z_n = a_{k_n}$ we have $f'(a) \in \mathbb{R}$. Now, $f_1(z) := \dfrac{f(z) - f(a)}{z - a}$ is analytic (w/ $f_1(a) = f'(a)$) and satisfies the property that $f_1(a_{k_n}) \in \mathbb{R}$. Hence, we may repeat the process, i.e. examine $f_2(z) := \dfrac{f_1(z) - f_1(a)}{z-a}$, etc.

The result is a sequence $f_n(z)$ of analytic functions satisfying

$$\begin{cases} f_{n-1}(z) = f_{n-1}(a) + f_n(z)(z-a) \\ \qquad f_n(a_{k_n}) \in \mathbb{R} \Rightarrow f_n(a) \in \mathbb{R} \end{cases}$$

$$\begin{aligned} \Rightarrow \quad f(z) &= f(a) + f_1(z)(z-a) \\ &= f(a) + f_1(a)(z-a) + f_2(z)(z-a)^2 \\ &= \cdots = f(a) + f_1(a)(z-a) + \cdots + f_{n-1}(a)(z-a)^{n-1} \\ &\qquad + f_n(z)(z-a)^n. \end{aligned}$$

Differentiating n times, we obtain $f^{(n)}(a) = n!\, f_n(a) \in \mathbb{R}$.

Since this is true for all n, we conclude $f(z) = \sum_{n=0} a_n (z-a)^n$, $a_n \in \mathbb{R}$, hence $f(\mathbb{R}) \subset \mathbb{R}$, as f is entire. $\square$

(4) Assume $f_n(z) = \sum_{k=0}^{n} \frac{1}{k! z^k}$, $z \neq 0$, and fix $\varepsilon > 0$. Show for sufficiently large n that all the zeroes of f_n are in the disk $D(0, \varepsilon)$.

$$f_n(w) = \sum_{k=0}^{n} \frac{w^k}{k!}, \quad w = \frac{1}{z}.$$

$f_n(w)$ converges uniformly to e^w on any disk $|w| \leq R$ (the convergence is uniform because this is a power series w/ infinite radius of convergence). Now, $|e^w| > 0$ on $|w| \leq R$, hence by continuity + compactness $|e^w| \geq \delta > 0$ on $|w| \leq R$.

By uniform convergence, for sufficiently large n, $|f_n(\omega)| \geq |e^\omega| - \frac{\delta}{2} \geq \frac{\delta}{2} > 0$, valid for $|\omega| \leq R$.

Translating back to z, take $R = \frac{1}{\varepsilon}$ above. Then for sufficiently large n, if $|z| \geq \varepsilon$, i.e. $|\omega| = \frac{1}{|z|} \leq \frac{1}{\varepsilon} = R$, we have $0 < |f_n(\omega)| = \left|\sum_{k=0}^{n} \frac{\omega^k}{k!}\right| = \left|\sum_{k=0}^{n} \frac{1}{k! z^k}\right| = |f_n(z)|$

Hence, the zeroes of $f_n(z)$ must lie in $|z| < \varepsilon$, as desired. $\square$