COMPLEX ANALYSIS GRADUATE EXAM
Fall 2011

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

(1) Evaluate

$$\int_0^{2\pi} \frac{d\theta}{3 + \cos\theta + 2\sin\theta}.$$

(2) Suppose the series $f(z) = \sum_{n=0}^{\infty} c_n z^n$ converges for $|z| < R$. Show that for $r < R$,

$$\int_{\{|z|=r\}} |f(z)|^2 \, dz = 2\pi \sum_{n=0}^{\infty} |c_n|^2 r^{2n}.$$

(3) Let $f(z)$ be analytic on $\mathbb{C}$ and suppose that the line $\Gamma = \{t + it : t \in \mathbb{R}\}$ is mapped to itself, that is, $f(z) \in \Gamma$ for all $z \in \Gamma$. If $f(\sqrt{2}) = 3$, then what is $f(\sqrt{2}i)$?

(4) Let $\Omega \subset \mathbb{C}$, with $\Omega \neq \mathbb{C}$, be simply connected, and let $f : \Omega \to \Omega$ be a conformal bijection. If f has two distinct fixed points z_1, z_2 (that is, $f(z_1) = z_1, f(z_2) = z_2$), show that f is the identity map.

COMPLEX FALL 2011

(1) Evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{3+\cos\theta+2\sin\theta}$

Examine $\displaystyle\frac{ie^{i\theta}}{ie^{i\theta}(3+\cos\theta+2\sin\theta)}$.

The bottom is $ie^{i\theta}\left(3+\frac{e^{i\theta}+e^{-i\theta}}{2}-i\left(e^{i\theta}-e^{-i\theta}\right)\right)$

$= 3ie^{i\theta} + ie^{i\theta}\left(e^{i\theta}\left(\tfrac{1}{2}-i\right)+e^{-i\theta}\left(\tfrac{1}{2}+i\right)\right)$

$= \underbrace{(\qquad)}_{-1+\frac{i}{2}} + (3i)z + \left(1+\tfrac{i}{2}\right)z^2 \overset{}{=} p(z) \quad (z=e^{i\theta}).$

The roots are $\displaystyle\frac{-3i \pm \sqrt{-9-4\left(1+\frac{i}{2}\right)\left(-1+\frac{i}{2}\right)}}{2\left(1+\frac{i}{2}\right)}$

Discriminant $= -9-(2+i)(-2+i) = -4$

roots $= \displaystyle\frac{-3i \pm 2i}{2+i} \begin{array}{l}\to \cancel{-5i} \\ \to -i \leftarrow \text{only root in unit circle.}\end{array}$

$2+i \leftarrow$ magnitude $\sqrt{5}$

$\displaystyle\frac{-i}{2+i}$

Since $\dfrac{1}{p(z)}$ has a pole of order 1

at $z = \dfrac{-i}{2+i}$, we have

$$\int_C^{2\pi} \frac{d\theta}{3+\cos\theta+2\sin\theta} = \int_{\{|z|=1\}} \frac{dz}{p(z)} = \frac{2\pi i}{p'\left(\frac{-i}{2+i}\right)}.$$

$$p'(z) = 3i + (2+i)z$$

$$\therefore \text{ integral } = \frac{2\pi i}{3i+(2+i)\left(\frac{-i}{2+i}\right)} = \frac{2\pi i}{2i} = \pi.$$

$\square$

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(2) $\quad f(z) = \displaystyle\sum_{n=0}^{\infty} c_n z^n \quad (|z|<R)$

Show $\displaystyle\int_{\{|z|=r\}} \frac{|f(z)|^2}{iz}\,dz = 2\pi \sum_{n=0}^{\infty} |c_n|^2 r^{2n}$

$(0<r<R)$

<u>calculation</u> $\gamma(t) = re^{it}$

$$\int_{\{|z|=r\}} \frac{|f(z)|^2}{iz} dz \overset{①}{=} \int_0^{2\pi} \frac{\gamma'(t)}{i\gamma(t)} \left(\sum_{n=0}^{\infty} C_n r^n e^{int}\right)\cdot\left(\sum_{m=0}^{\infty} \overline{C_m} r^m e^{-imt}\right) dt$$

$$\overset{②}{=} \sum_{i=0}^{\infty} \sum_{n+m=i} r^{n+m} C_n \overline{C_m} \int_0^{2\pi} e^{i(n-m)t} dt$$

$$\underset{n=m}{\overset{③}{=}} 2\pi \sum_{n=0}^{\infty} |C_n|^2 r^{2n} \quad ✓$$

To justify ① : <u>Since</u> conjugation is linear (and continuous) $\overline{\sum_{n=0}^{\infty} C_n z^n} = \sum_{n=0}^{\infty} \overline{C_n}\, \overline{z}^n$.

To justify ②: we NTS

$$\sum_{i=0}^{\infty} \int_0^{2\pi} \sum_{n+m=i} r^{n+m} C_n \overline{C_m} e^{i(n-m)t} dt = \int_0^{2\pi} \sum_{i=0}^{\infty} \sum_{n+m=i} r^{n+m} C_n \overline{C_m} e^{i(n-m)t} dt$$

It suffices to show $\infty > \sum_{i=0}^{\infty} \int_0^{2\pi} \left| \sum_{n+m=i} r^{n+m} C_n \overline{C_m} e^{i(n-m)t} \right| dt$

$$= \sum_{i=0}^{\infty} \int_0^{2\pi} \sum_{n+m=i} r^{n+m} |C_n||C_m| \overset{\text{positivity}}{=} \int_0^{2\pi} \sum_{i=0}^{\infty} \sum_{n+m=i} r^{n+m} |C_n||C_m| dt$$

$$= 2\pi \left(\sum_{n=0}^{\infty} |C_n| r^n \right)^2 \overset{?}{<} \infty$$

Since $\sum_{n=0}^{\infty} C_n z^n$ converges for $|z| = r$,

it converges absolutely at $z = r$,

being a power series w/ $r <$ radius of convergence

i.e. $2\pi \left(\sum_{n=0}^{\infty} |C_n| r^n \right)^2 < \infty$ ✓

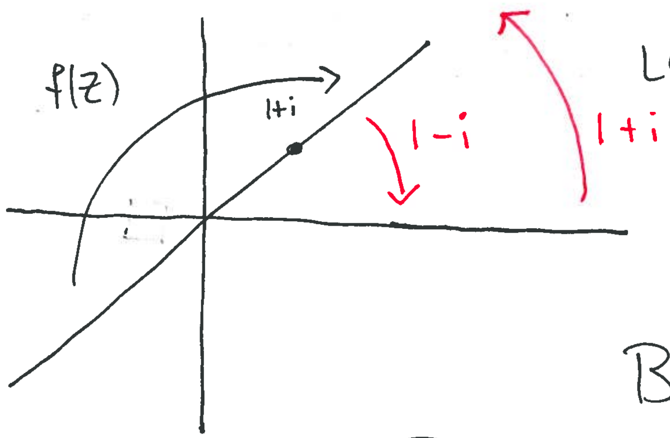
and we can switch the integral & summation.

I forgot to mention that we can multiply the series as such also because of absolute convergence of $\sum_{n=0}^{\infty} C_n z^n$, $|z| < R$.

To justify (3): This is just Cauchy's theorem when $n \neq m$.

$\square$

(3). $f(z)$ analytic on $\mathbb{C}$ and maps the line $\{t + it : t \in \mathbb{R}\}$ to itself. If $f(\sqrt{2}) = 3$, what is $f(\sqrt{2}\, i)$?



Let $g(z) = (1-i) f(z(1+i))$.

Then $g(z)$ takes the real axis to itself.

By the Schwarz Reflection Principle, $g(z)\big|_{\operatorname{im} z \geq 0}$ extends across the real axis to an entire function g', $\overline{g(z)} = g'(\bar{z})$. But since $g(z)$ is entire and agrees w/ the extended function on $\{z : \operatorname{im} z \geq 0\}$, the two functions agree, i.e. $g(\bar{z}) = \overline{g(z)}$ for all $z \in \mathbb{C}$.

Solve $w(1+i) = \sqrt{2}$

$$w = \frac{1-i}{2}\sqrt{2} = \frac{1-i}{\sqrt{2}}.$$

$g(\omega) = (1-i)3$. $\overline{g(\omega)} = (1+i)3$.

observe $(1+i)\overline{\omega} = (1+i)\left(\frac{1+i}{\sqrt{2}}\right) = \frac{+2i}{\sqrt{2}} = +\sqrt{2}i$

so $g(\overline{\omega}) = (1-i)f(\sqrt{2}i) = \overline{g(\omega)} = 3(1+i)$.

Hence, $f(\sqrt{2}i) = \frac{1+i}{2}\,3(1+i) = 3i$.

$\square$

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(4) $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$,

Ω simply connected.

Let $f: \Omega \to \Omega$ be a bijective conformal mapping s.t. $f(z_1) = z_1$ and $f(z_2) = z_2$, $z_1 \neq z_2$.

Show f is the identity.

First assume $\Omega = D = \{z : |z| < 1\}$.

We know that any conformal bijection from the unit disk to itself is a Möbius transformation of the form

$$S(z) = e^{i\theta} \frac{z - \alpha}{1 - \bar{\alpha} z}, \quad |\alpha| < 1.$$

We show that either there is a single fixed point w/ multiplicity one inside the disk or the two fixed points possibly a single fixed point w/ multiplicity 2 lie on $\{z : |z| = 1\}$. (Remember Möbius transformations always have 2 fixed points up to multiplicity.)

When $\alpha = 0$, $S(z) = e^{i\theta} z$ is a rotation, which indeed has a fixed point at $z = 0$ & $z = \infty$.

Otherwise $\alpha \neq 0$. Consider $e^{i\theta}(z - \alpha) = z(1 - \bar{\alpha} z)$.

$\Rightarrow e^{i\theta} z - \alpha e^{i\theta} = z - \bar{\alpha} z^2 \Rightarrow$

$\bar{\alpha} z^2 + (e^{i\theta} - 1) z - \alpha e^{i\theta}$. The roots z_1, z_2 are the fixed points. Observe

$$z_1 z_2 = \frac{(e^{i\theta} - 1)^2 - \left((e^{i\theta} - 1)^2 + 4|\alpha|^2 e^{i\theta}\right)}{4\bar{\alpha}^2}$$

$$= -\frac{\alpha}{\bar{\alpha}} e^{i\theta}$$

$\Rightarrow |z_1||z_2| = 1$. So either $|z_1| < 1$ and $|z_2| > 1$

implying there is a single fixed point of multiplicity one inside the disk, or $|z_1| = |z_2| = 1$ and all the fixed points lie on the boundary, as claimed. This proves the result for the disk.

The general case follows immediately by the Riemann mapping theorem: If $\Omega \xrightarrow{f} \Omega$ fixes z_1 & z_2 and $\Omega \xrightarrow{g} D$ is a conformal bijection setting $w_1 = g(z_1) \neq g(z_2) = w_2$ we have that $g \circ f \circ g^{-1} : D \to D$ is a conformal bijection sending $w_1 \to z_1 \to z_1 \to w_1$ and similarly for w_2, violating what we proved above. $\square$