COMPLEX ANALYSIS GRADUATE EXAM
Spring 2010

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

Notation: $\Re z$ denotes the real part of the complex number z, and $\Im z$ its imaginary part.

(1) Map the region $\Omega = \{\Im z > 0\} \setminus \{iy : 0 < y \leq 1\}$ conformally to the unit disk $D = \{|z| < 1\}$.

(2) How many zeroes of $p(z) = z^4 + z^3 + 4z^2 + 2z + 7$ lie in the right half plane $\{\Re z > 0\}$?

(3) Let f be analytic in the unit disk $D = \{|z| < 1\}$ and continuous on its closure $\bar{D}$. Show that if f is real valued on the boundary $\partial D = \{|z| = 1\}$ then f must be a constant.

(4) By consideration of $\int e^{z+\frac{1}{z}}\, dz$, or otherwise, show that

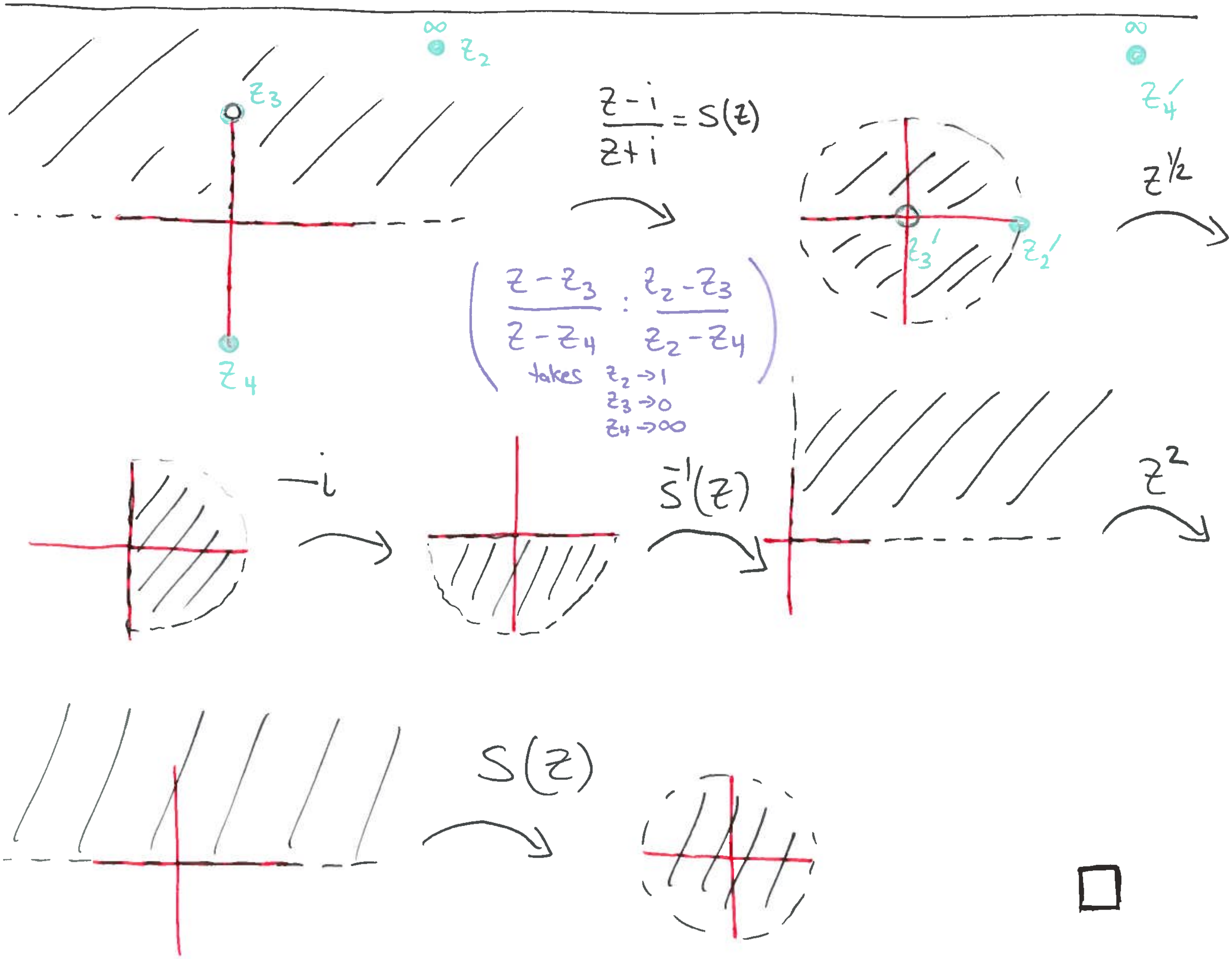
$$\frac{1}{2\pi}\int_0^{2\pi} e^{2\cos\theta}\cos\theta\, d\theta = 1 + \frac{1}{2!} + \frac{1}{2!3!} + \frac{1}{3!4!} + \cdots .$$

Complex, Spring 2010

(1) Exhibit a conformal mapping from

$$\Omega = \{\operatorname{Im} z > 0\} - \{iy : 0 < y \le 1\}$$

to the unit disk $D = \{z : |z| < 1\}$.



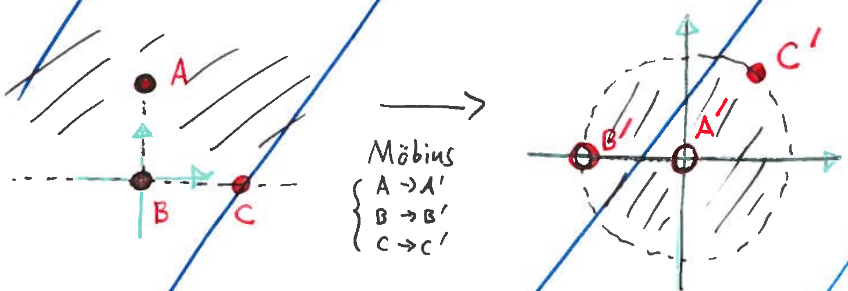
$\square$

Complex, Spring 2010

1) Exhibit a conformal mapping from

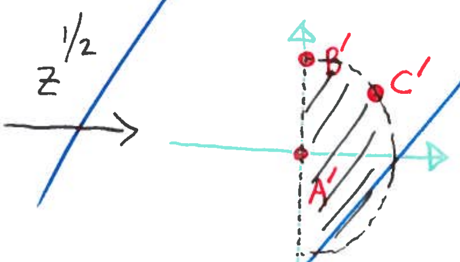
$$\Omega = \{\operatorname{im} z > 0\} - \{iy : 0 < y \leq 1\}$$

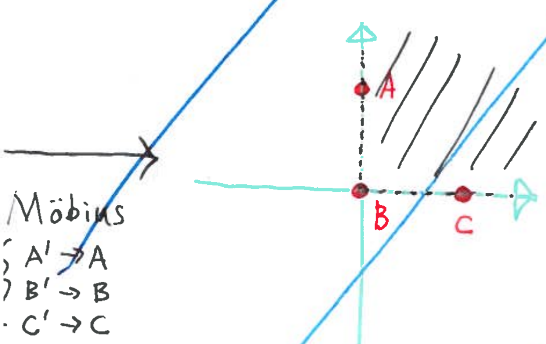
to the unit disk $D = \{z : |z| < 1\}$.



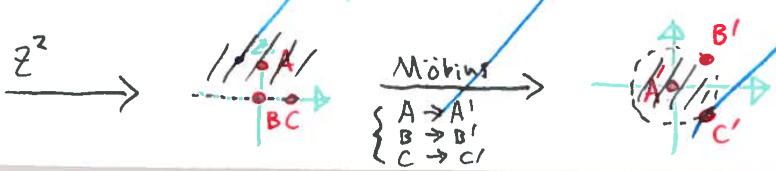
Here, $\overset{\cdots}{AB} \mapsto \overset{\cdots}{A'B'}$

Since the line $\overleftrightarrow{AB} \mapsto \overleftrightarrow{A'B'}$ because it can't go to a circle, as this would be entirely contained in the disk, whereas $\overleftrightarrow{AB}$ is not entirely contained in the upper half plane.





Here, $\overleftrightarrow{A'B'} \mapsto \overleftrightarrow{AB}$ for the same reason as before, and by connectedness the disk splits as shown.



□

(2) How many zeroes lie in the right half plane $\{z : \operatorname{re} z > 0\}$ of the polynomial

$p(z) = z^4 + z^3 + 4z^2 + 2z + 7$?

We use Rouché's theorem, which says that if $\gamma \sim 0$ and $n(\gamma, z) = 0$ or 1 for all $z \notin \gamma$, and if $f(z)$ & $g(z)$ are analytic w/ $|f(z) - g(z)| < |f(z)|$ on γ, then $f(z)$ & $g(z)$ have the same number of zeroes enclosed by γ.

Set $f(z) = z^4 + z^3 + 4z^2 + 2z + 7$

and $g(z) = z^4 + 3$.

On $\{z = bi : b \in \mathbb{R}\}$, $f(z) = (b^4 - 4 + 7) + i(-b^3 + 2b)$

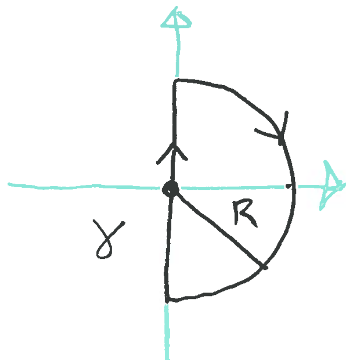
Since $\operatorname{Re}(f(z) - g(z)) = 0 < \operatorname{Re} f(z)$

and $\operatorname{Im}(f(z) - g(z)) = \operatorname{Im} f(z)$,

it is clear that

$|f(z) - g(z)| < |f(z)|$ on $\{z = bi : b \in \mathbb{R}\}$

If we let γ be the following path,



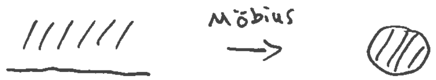
since $f(z) - g(z)$ is a 3rd degree polynomial and $f(z)$ is a 4^{th} degree polynomial, it is clear that for R sufficiently large

$|f(z) - g(z)| < |f(z)|$ on the arc. Since this is also true on the imaginary axis, we conclude by Rouche's theorem that $f(z)$ and z^4+3 have the same number of zeroes in the right half plane, namely, two $\left(\underline{\underline{\underline{\frac{\sqrt{2}}{2} \pm i\frac{\sqrt{2}}{2}}}}, -\frac{\sqrt{2}}{2} \pm i\frac{\sqrt{2}}{2} \right)$. $\square$

---

(3) f analytic on unit disk and continuous on boundary and real-valued on boundary. Show f is constant.

By precomposing w/ a Möbius transformation taking the closed upper half plane to the closed unit disk



we obtain an analytic function $\overline{H} \to \mathbb{C}$. Since the original function was continuous on $\overline{D}$, it is bounded, hence the function $\overline{H} \to \mathbb{C}$ is bounded. This function is real on the real axis, since the original function was real on ∂D. By the Schwarz Reflection Principle, we may extend our function $\overline{H} \to \mathbb{C}$ to an entire function by the formula $f(\bar{z}) = \overline{f(z)}$. The result is a bounded entire function, which must therefore be constant, by Liouville's theorem. So the original function must have been constant. $\square$

---

(4) By consideration of $\int e^{z+\frac{1}{z}} dz$ show that

$$\frac{1}{2\pi}\int_0^{2\pi} e^{2\cos\theta}\cos\theta \, d\theta = \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!}.$$

Letting $\gamma(t) = e^{it}$, we have $\int_\gamma e^{z+\frac{1}{z}} dz = \int_0^{2\pi} e^{e^{it}+e^{-it}} ie^{it} dt$

$= \int_0^{2\pi} e^{2\cos t} ie^{it} dt$, whose imaginary part is $\int_0^{2\pi} e^{2\cos t}\cos t \, dt$.

By the Residue Theorem, the integral $\int_\gamma e^{z+\frac{1}{z}} dz$ is equal to

$2\pi i \,\mathrm{Res}(e^{z+\frac{1}{z}}, 0)$. Since the series for e^z and $e^{1/z}$ converge absolutely, we have

$$e^z e^{\frac{1}{z}} = \left(\sum_{n=0}^{\infty} \frac{z^n}{n!}\right)\left(\sum_{m=0}^{\infty} \frac{1}{z^m m!}\right) = \sum_{n,m=0}^{\infty} \frac{1}{z^{m-n}\, n!\, m!}.$$

The coefficient of $\frac{1}{z}$ in this Laurent expansion is therefore $\sum_{n=0}^{\infty} \frac{1}{n!(n+1)!}$. We conclude by taking the imaginary part that

$$2\pi \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!} = \mathrm{Im} \int_{\gamma} e^{z+\frac{1}{z}}\, dz$$

$$= \int_0^{2\pi} e^{2\cos\theta} \cos\theta \, d\theta. \qquad \square$$