COMPLEX ANALYSIS GRADUATE EXAM

Fall 2010

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Show that
$$\int_0^\infty \frac{\sin x}{x(x^2+1)}\,dx = \frac{\pi(1-e^{-1})}{2}.$$

2. Suppose that f is holomorphic in a neighborhood of 0 and that

$$(*) \qquad \sum_{n=0}^{\infty} f^{(n)}(z)$$

is absolutely convergent at $z=0$. Show that f is an entire function, and that $(*)$ is convergent for all $z \in \mathbb{C}$.

3. Let f be a non-negative real valued harmonic function in the disc $D = \{z \in \mathbb{C} : |z| < R\}$.

(i) Prove that

$$\frac{R-|z|}{R+|z|} f(0) \le f(z) \le \frac{R+|z|}{R-|z|} f(0) \quad \text{whenever } |z| < R.$$

[Hint: use the Poisson formula.]

(ii) Prove that
$$\frac{1}{3} f(0) \le f(z) \le 3f(0) \quad \text{whenever } |z| \le R/2.$$

(iii) Let K be a compact subset of the open disc D. Show that there is a constant M depending only on K and R such that

$$f(z_1) \le M f(z_2) \quad \text{for all } z_1, z_2 \in K.$$

4. Liouville's theorem states that a bounded entire function f is constant.

i) Give a proof of Liouville's theorem. You may use standard results about holomorphic functions such as Cauchy's theorem and power series representation, but any result you use should be clearly stated

(ii) Suppose instead that f is entire and that $|f(z)| \le K(1+|z|^n)$ for some $K < \infty$ and positive integer n. Show that f is a polynomial of degree at most n.

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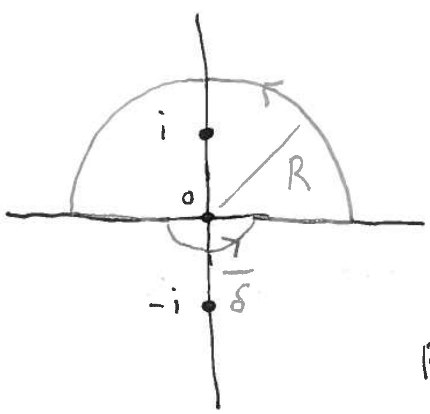
1) Show $\displaystyle\int_0^\infty \frac{\sin x}{x(x^2+1)}\,dx = \frac{\pi(1-\frac{1}{e})}{2}$.

Consider $\displaystyle\int \frac{e^{iz}}{z(z^2+1)}$. Let $\gamma_{R,\delta}$ be the path shown below. Then standard arguments



using the Residue Theorem show (by taking the limit as $R\to\infty$ and $\delta\to 0$)

$$2\pi i \,\mathrm{Res}(i) + \pi i \,\mathrm{Res}(0) = \lim_{\delta\to 0} \int_\delta^\infty + \int_{-\infty}^{-\delta} \frac{e^{ix}}{x(x^2+1)}\,dx.$$

$$\mathrm{Res}(i) = \frac{e^{-1}}{i(2i)} = \frac{-1}{2e}.$$

$$\mathrm{Res}(0) = \frac{1}{i(-i)} = 1.$$

The imaginary part of the expression on the right is

$$\lim_{\delta\to 0} \int_\delta^\infty + \int_{-\infty}^{-\delta} \frac{\sin x}{x(x^2+1)}\,dx.$$

Since this limit exists and the integrand is even, we have (again taking imaginary parts) $-\frac{2\pi}{2e} + \pi = 2\displaystyle\int_0^\infty \frac{\sin x}{x(x^2+1)}\,dx \Rightarrow \int_0^\infty \frac{\sin x}{x(x^2+1)}\,dx = \frac{\pi}{2}\left(1-\frac{1}{e}\right).$ □

2) f entire and $\sum_{n=0}^{\infty} f^{(n)}(z)$ is absolutely convergent at $z=0$. Show this series converges for all z.

Locally around $z=0$, f has the power series $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$. Since f is entire, this power series converges for all z and equals $f(z)$.

Now, $f^{(n)}(z) = \sum_{k=n}^{\infty} f^{(k)}(0) \frac{z^{k-n}}{(k-n)!}$.

So $|f^{(n)}(z)| \leq \sum_{k=n}^{\infty} |f^{(k)}(0)| \frac{R^{k-n}}{(k-n)!}$ where $R=|z|$.

So $\sum_{n=1}^{\infty} |f^{(n)}(z)| \leq \sum_{k=1}^{\infty}\sum_{n=1}^{k} |f^{(k)}(0)| \frac{R^{k-n}}{(k-n)!}$

$$= \sum_{k=1}^{\infty} |f^{(k)}(0)| \sum_{n=1}^{k} \frac{R^{k-n}}{(k-n)!} \leq \sum_{k=1}^{\infty} |f^{(k)}(0)| \sum_{n=0}^{k-1} \frac{R^n}{n!}$$

$$\leq \sum_{k=1}^{\infty} |f^{(k)}(0)| e^R = e^R \sum_{k=1}^{\infty} |f^{(k)}(0)|$$

$< \infty$,

where the reverse of summation is justified because all terms are positive.

we conclude $\sum_{n=1}^{\infty} f^{(n)}(z)$ converges for all z.

Remark: we could have assumed f is analytic on a small disk $\{z : |z| < \delta\}$ and used the hypotheses to show f extends to the analytic function $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$ on all of $\mathbb{C}$. To see convergence of this series, observe $\sum_{n=0}^{\infty} |f^{(n)}(0)| \frac{R^n}{n!}$

$$\leq \sum_{n=0}^{\infty} |f^{(n)}(0)| e^R = e^R \sum_{n=0}^{\infty} |f^{(n)}(0)| < \infty .$$

The two functions coincide on $\{z : |z| < \delta\}$ since they agree on a disk. $\square$

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3) f nonnegative harmonic on $D = \{z : |z| < R\}$.
(we also should assume f is continuous on $\{z : |z| = R\}$.)

(i) Show $\frac{R-|z|}{R+|z|} f(0) \leq f(z) \leq \frac{R+|z|}{R-|z|} f(0) \quad (|z|<R)$.

We use Poisson's formula: $f(z) = \frac{1}{2\pi} \int_{|\xi|=R} \frac{R^2 - |z|^2}{|\xi - z|^2} f(\xi)\, d\theta \quad (|z|<R)$.

$$|\xi - z| \leq |\xi| + |z| = R + |z| \implies \frac{(R+|z|)(R-|z|)}{|\xi - z|^2} \geq \frac{R-|z|}{R+|z|}.$$

$$|\xi - z| \geq |\xi| - |z| = R - |z| \implies \frac{(R+|z|)(R-|z|)}{|\xi - z|^2} \leq \frac{R+|z|}{R-|z|}.$$

Since $f(0) = \frac{1}{2\pi} \int_{|\xi|=R} f(\xi) d\theta$, the result follows. $\square$

(ii) Show $\frac{1}{3} f(0) \leq f(z) \leq 3 f(0) \quad \left(|z| \leq \frac{R}{2} \right)$.

Using the previous part, the result follows since

$$|z| \leq \frac{R}{2} \Rightarrow \frac{R+|z|}{R-|z|} \leq \frac{\frac{3R}{2}}{\frac{R}{2}} = 3 \quad \&$$

$$|z| \leq \frac{R}{2} \Rightarrow \frac{R-|z|}{R+|z|} \geq \frac{\frac{R}{2}}{\frac{3R}{2}} = \frac{1}{3}. \quad \square$$

(iii) K compact. $K \subset D$. Show there is a constant $M = M(K, R)$ s.t. $f(z_1) \leq M f(z_2) \quad z_1, z_2 \in K$

Let $\alpha < R$ be s.t. $K \subset \{ z : |z| \leq \alpha \}$.
Then for $z_1, z_2 \in K$, by part (i)

$$f(z_1) \leq \frac{R+\alpha}{R-\alpha} f(0) \quad \& \quad f(z_2) \geq \frac{R-\alpha}{R+\alpha} f(0)$$

$$\Longrightarrow \quad f(z_1) \leq \underbrace{\frac{R+\alpha}{R-\alpha} \frac{R+\alpha}{R-\alpha}}_{\equiv M} f(z_2) \quad \square$$

(4) Prove if f is entire and

$$|f(z)| \leq K(1 + |z|^n) \quad \text{for all } z$$

where K is a constant and $n \geqslant 0$, then $f(z)$ is a polynomial of degree at most n.

(Liouville's theorem is a special case, when $n = 0$.)

Since f is entire, for any $R > 0$ we have the representation

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{|\xi - z| = R} \frac{f(\xi)}{(z - \xi)^{n+1}} \, d\xi .$$

$$|f^{(n+1)}(z)| \leq \frac{(n+1)!}{2\pi} \int_{|\xi - z| = R} \frac{K(1 + |\xi|^n)}{R^{n+2}} \, d\xi$$

$|\xi - z| = R$

$|\xi| = |\xi + z - z| \leq |\xi - z| + |z|$

$$\leq \frac{(n+1)!}{2\pi} \frac{K(1 + (R + |z|)^n)}{R^{n+2}} 2\pi R$$

$$= \frac{(n+1)! \, K(1 + (R + |z|)^n)}{R^{n+1}}$$

$\longrightarrow 0$ as $R \longrightarrow \infty$ since $|z|$ is fixed, hence $f^{(n+1)}(z) = 0$ for all z.

The conclusion is that $f(z)$ is a polynomial of degree at most n. $\square$