MATH 505a Spring 2022 Qual Solution Attempts

Troy Tao

August 4, 2022

Contact yntao@usc.edu if you think this document needs revision.

Problem 1

(a)

Let X_1, X_2, X_3 be independent exponential random variables with parameter $\lambda = 1$. So $\mathbb{P}(X_i > x) = e^{-x}, x > 0$. Find

$$\mathbb{E}\left(\frac{X_1}{X_1 + X_2 + X_3}\right).$$

Solution. Let $Y = X_1 + X_2$, then compute the pdf of Y:

$$\begin{aligned}
\mathbb{P}(X_2 + X_3 \leq y) &= \int_0^y \mathbb{P}(X_2 \leq y - x) f_{X_3}(x)dx \\
&= \int_0^y (1 - e^{-(y-x)})e^{-x}dx \\
&= \left[-e^{-x} - xe^{-x}\right]_0^y \\
&= 1 - e^{-y}(y+1),\ y \geq 0
\end{aligned}$$

So by differentiating the cdf, we have

$$f_Y(y) = ye^{-y},\ y \geq 0$$

Now consider the probability:

$$\begin{aligned}
\mathbb{P}\left(\frac{X_1}{X_1 + Y} \leq z\right) &= \mathbb{P}\left(X_1 \leq \frac{zY}{1-z}\right) \\
&= \int_0^\infty \mathbb{P}\left(X_1 \leq \frac{zy}{1-z}\right) f_Y(y)dy \\
&= \int_0^\infty \left[1 - exp\left(-\frac{zy}{1-z}\right)\right] \cdot y \cdot exp(-y)dy \\
&= 1 - (1-z)^2,\ 0 \leq z \leq 1
\end{aligned}$$

And from the fact that $\frac{X_1}{X_1+Y} \geq 0$ (only takes non-negative value), we can compute the expectation by the complementary cdf:

$$\mathbb{E}\left(\frac{X_1}{X_1 + X_2 + X_3}\right) = \int_0^1 (1-z)^2 dz = \frac{1}{3}$$

(b)

Let (X, Y) be independent uniforms on $[0, 1]$. Find the joint density function of X and $V = X + Y$. Find $f(x|v)$, the density function of X conditional on $V = v$. Also, find $\mathbb{E}(X|V)$.

Proof. First we should compute the pdf of V. When $0 < v < 1$:

$$\mathbb{P}(X + Y \leq v) = \int_0^v (v - y)dy = \frac{1}{2}v^2$$

When $1 \leq v < 2$:

$$\mathbb{P}(X + Y \leq v) = \int_{v-1}^1 (v - y)dy + (v - 1) = -\frac{1}{2}v^2 + 2v - 1$$

So we have the pdf:

$$f_V(v) = \begin{cases} v & 0 < v < 1 \\ 2 - v & 1 \leq v < 2 \end{cases}$$

Note that the conditional pdf is

$$f_{X|V}(x|v) = \frac{f_{X,V}(x, v)}{f_V(v)} = \frac{f_X(x)f_Y(y)}{f_V(v)}, \ y = v - x$$

plug in the previous results,

$$f_{X|V}(x|v) = \begin{cases} \frac{1}{v} & 0 < x < v < 1 \\ \frac{1}{2-v} & 1 \leq v < 2, \ v - 1 < x < 1 \end{cases}$$

So the expectation follows:

$$\mathbb{E}(X|V) = \frac{1}{2}V \cdot \mathbb{1}_{(0,1)}(V) + \frac{2V - V^2}{4 - 2V} \cdot \mathbb{1}_{[1,2)}(V)$$

□

Problem 2

In an election, candidates A receives n votes, and candidate B receives m votes, where $n > m$. Assuming that all $\binom{n+m}{m}$ orderings are equally likely, show that the probability that A is always ahead in the count of votes is $(n - m)/(n + m)$.

Proof. Denote $P_{i,j} = \mathbb{P}(\text{A is always ahead} \mid \text{A received } i \text{ votes, B received } j \text{ votes})$. By conditioning on the last vote, we have

$$\begin{aligned} P_{n,m} &= \mathbb{P}(\text{A received the last vote})P_{n-1,m} + \mathbb{P}(\text{B received the last vote})P_{n,m-1} \\ &= \frac{n}{n+m}P_{n-1,m} + \frac{m}{n+m}P_{n,m-1} \end{aligned}$$

Then we construct an induction: $\forall k \in \mathbb{N}$, if $n + m = k$ and $n > m$, then $P_{n,m} = \frac{n-m}{n+m}$.

1. $k = 1,\ n > m \implies n = 1,\ m = 0$ and $P_{1,0} = 1$

2. Given that the statement is true for $n + m = k$, let $n + m = k + 1,\ n > m$. Then we have

$$P_{n,m} = \frac{n}{n+m} P_{n-1,m} + \frac{m}{n+m} P_{n,m-1}$$

Note that $n + m - 1 = k \implies P_{n,m-1} = \frac{n-m+1}{n+m-1}$ and $P_{n-1,m} = \frac{n-m-1}{n+m-1}$. So

$$\begin{aligned} P_{n,m} &= \frac{n}{n+m}\frac{n-1-m}{n-1+m} + \frac{m}{n+m}\frac{n-m+1}{n+m-1} \\ &= \frac{(n-1+m)(n-m)}{(n-1+m)(n+m)} \\ &= \frac{n-m}{n+m} \end{aligned}$$

In the case that $n = m+1$, $P_{n-1,m} = 0$ since B eventually will catch up A. Therefore, although it's not in our assumption, the equation $P_{n-1,m} = 0 = (n-1-m)/(n+m-1)$ is still true.

□

Problem 3

Let n be a positive integer with prime factorization $n = p_1^{m_1} \cdots p_k^{m_k}$ for distinct primes $p_1, \cdots, p_k$ with $m_1, \cdots, m_k > 0$. Choose an integer N uniformly at random from the set $\{1, 2, \cdots, n\}$.. Show that the probability that N shares no common prime factor with n is equal to

$$\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right).$$

(Hint: use inclusion-exclusion)

Proof. Let $A_i = \{N = ap_i, a \in \mathbb{N}\}$ and for $I \subset \{1, 2, \cdots, n\}$

$$\begin{aligned} \mathbb{P}(\cap_{i\in I} A_i) &= \mathbb{P}(N = a\prod_{i\in I} p_i,\ a \in \mathbb{N}) \\ &= \frac{\frac{n}{\prod_{i\in I} p_i}}{n} \\ &= \prod_{i\in I} \frac{1}{p_i} \end{aligned}$$

Now by inclusion-exclusion theorem,

$$
\begin{aligned}
\mathbb{P}(\text{N is co-prime to n}) &= 1 - \mathbb{P}(\cup_{i=1}^{k} A_i) \\
&= 1 - \left(\sum_{i=1}^{k} \frac{1}{p_i} - \sum_{i,j=1}^{k} \frac{1}{p_i p_j} + \cdots + (-1)^{k+1} \frac{1}{\prod_{i=1}^{k} p_i} \right) \\
&= 1 + \sum_{i=1}^{k} \left(-\frac{1}{p_i} \right) + \sum_{i,j=1}^{k} \left(-\frac{1}{p_i} \right) \left(-\frac{1}{p_j} \right) + \cdots + \prod_{i=1}^{k} \left(-\frac{1}{p_i} \right) \\
&= \left(1 - \frac{1}{p_1} \right) \left(1 - \frac{1}{p_2} \right) \cdots \left(1 - \frac{1}{p_k} \right)
\end{aligned}
$$

□