MATH 505a Spring 2021 Qual Solution Attempts

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Problem 1

A permutation π on n symbols is said to have i as a fixed point if $\pi(i) = i$.

(a)

Find the probability p_n that a random permutation of n symbols has no fixed points. HINT: Principle of inclusion and exclusion. (Your answer may involve a finite sum, which you don't need to simplify.)

Solution. Use inclusion-exclusion: let A_i be the set of permutations that $\pi(i) = i$.

$$\begin{aligned} p_n &= 1 - \mathbb{P}(\cup_{i=1}^n A_i) \\ &= 1 - (\sum_i \mathbb{P}(A_i) - \sum_{i,j} \mathbb{P}(A_i \cap A_j) + \cdots + (-1)^{n+1}\mathbb{P}(\cap_{i=1}^n A_i) \\ &= 1 - n \cdot \frac{(n-1)!}{n!} + \binom{n}{2} \cdot \frac{(n-2)!}{n!} + \cdots + (-1)^n \binom{n}{n} \cdot \frac{1}{n!} \\ &= \sum_{p=0}^n (-1)^p \frac{1}{p!} \end{aligned}$$

(b)

Let S be a subset of $\{1, 2, \cdots, n\}$ of size k. Find the probability that the set of fixed points of a random permutation on n symbols is equal to S, and find the probability that a permutation has exactly k fixed points. HINT: If you didn't find the values p_j in part(a), you can still give answers for (b) expressed in terms of one or more p_j's.

Solution.

$$\begin{aligned}\mathbb{P}(\{\text{fixed points}\} = S) &= \mathbb{P}(\pi(i) = i,\ \forall i \in S) \cdot \mathbb{P}(\pi(j) \neq j,\ \forall j \in S^c | \pi(i) = i,\ \forall i \in S) \\ &= \frac{(n-k)!}{n!} p_{n-k} \\ \mathbb{P}(\text{k fixed points}) &= \binom{n}{k} \cdot \mathbb{P}(\{\text{fixed points}\} = S) \\ &= \frac{p_{n-k}}{k!}\end{aligned}$$

We get the second probability knowing that there are $\binom{n}{k}$ many sets with k fixed points.

(c)

Show that as n tends to infinity, the distribution of the number of fixed points converges to a Poisson(1) distribution.

Proof.

$$\begin{aligned}\lim_{n\to\infty} \mathbb{P}(\text{k fixed points}) &= \frac{1}{k!} \lim_{n\to\infty} \sum_{p=0}^{n-k} \frac{(-1)^p}{p!} \\ &= \frac{1}{k!} \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} \\ &= \frac{e^{-1}}{k!} \\ &\sim Poisson(1)\end{aligned}$$

□

Problem 2

Let $\{S_n,\ n \geq 0\}$ be symmetric simple random walk, that is, $S_n = \sum_{i=1}^{n} \xi_i$ with $\xi_1, \xi_2, \cdots$ i.i.d. satisfying $\mathbb{P}(\xi_1 = 1) = \mathbb{P}(\xi_1 = -1) = \frac{1}{2}$. Let $T = min\{n : S_n = 0\}$, and write $\mathbb{P}_a$ for probabilities when the walk starts at $S_0 = a$. By *basic probabilities* for $\{S_n\}$ we mean probabilities of the form $\mathbb{P}_0(S_n = k)$, $\mathbb{P}_0(S_n \geq k)$, or $\mathbb{P}_0(S_n \leq k)$, all of which corresponding to starting at $S_0 = 0$.

(a)

For $a \geq 1$, $i \geq 1$, $n \geq 1$, express $\mathbb{P}_a(S_n = i, T \leq n)$ and $\mathbb{P}_a(S_n = i, T > n)$ in terms of finitely many basic probabilities. HINT: Reflection principle.

Solution. Use reflection(reflect the part of path after the first approach at 0, with respect to 0), we have:

$$\mathbb{P}_a(S_n = i,\ T \leq n) = \mathbb{P}_a(S_n = -i) = \mathbb{P}_0(S_n = i + a)$$

Use conditional probability, we have:

$$\begin{aligned}
\mathbb{P}_a(S_n = i, T > n) &= \mathbb{P}_a(S_n = i) \cdot \mathbb{P}_a(T > n | S_n = i) \\
&= \mathbb{P}_a(S_n = i) \cdot (1 - \mathbb{P}_a(T \leq n | S_n = i)) \\
&= \mathbb{P}_a(S_n = i) \cdot (1 - \frac{\mathbb{P}_a(T \leq n, S_n = i)}{\mathbb{P}_a(S_n = i)}) \\
&= \mathbb{P}_a(S_n = i) - \mathbb{P}_a(S_n = i, T \leq n) \\
&= \mathbb{P}_0(S_n = i - a) - \mathbb{P}_0(S_n = i + a)
\end{aligned}$$

(b)

For $a \geq 1$, $i \geq 1$, $n \geq 1$, show that

$$\mathbb{P}_a(T > n) = \sum_{j=1-a}^{a} \mathbb{P}_0(S_n = j).$$

HINT: use (a) and look for cancellation

Proof.

$$\begin{aligned}
\mathbb{P}_a(T > n) &= \sum_{i=1}^{a+n} \mathbb{P}_a(S_n = i,\ T > n) \\
&= \sum_{i=1}^{a+n} \mathbb{P}_0(S_n = i - a) - \mathbb{P}_0(S_n = i + a) \\
&= \sum_{i=1-a}^{n} \mathbb{P}_0(S_n = i) - \sum_{j=1+a}^{n} \mathbb{P}_0(S_n = j) \\
&= \sum_{j=1-a}^{a} \mathbb{P}_0(S_n = j)
\end{aligned}$$

□

(c)

You may take as given that $\mathbb{P}_0(S_{2m} = 2j) \sim 1/\sqrt{\pi m}$ as $m \to \infty$ for each fixed $j \in \mathbb{Z}$; here $\sim$ means that ratio converges to 1. Use this to find $c,\ \alpha$ such that $\mathbb{P}_a(T > n) \sim c/n^\alpha$ as $n \to \infty$, where $a > 0$. Does c or α depend on a? HINT: It's enough to consider even n - why?

Proof. Assume n is even where $n = 2m$. For very large n, we have:

$$\begin{aligned}
\mathbb{P}_a(T > 2m) &= \sum_{j=1-a}^{a} \mathbb{P}_0(S_{2m} = j) \\
&= \sum_{j \in A} \mathbb{P}_0(S_{2m} = j), \ A = \{\text{even numbers in } \{1-a, 2-a, \cdots, a\}\} \\
&\sim a \cdot \frac{1}{\sqrt{\pi m}} \\
&= \frac{a\sqrt{\frac{2}{\pi}}}{n^{1/2}}
\end{aligned}$$

So we get $c = a\sqrt{\frac{2}{\pi}}$ and $\alpha = \frac{1}{2}$, where c depends on a, α does not.

Now we assume n is odd, and we will prove the convergence by squeezing. First by inclusion, we have the inequality:

$$\mathbb{P}_a(T > n-1) \geq \mathbb{P}_a(T > n) \geq \mathbb{P}_a(T > n+1)$$

divide the expected limit:

$$\frac{\mathbb{P}_a(T > n-1)}{c/n^\alpha} \geq \frac{\mathbb{P}_a(T > n)}{c/n^\alpha} \geq \frac{\mathbb{P}_a(T > n+1)}{c/n^\alpha}$$

normalize both sides:

$$\frac{\mathbb{P}_a(T > n-1)}{c/(n-1)^\alpha} \cdot \left(\frac{n}{n-1}\right)^\alpha \geq \frac{\mathbb{P}_a(T > n)}{c/n^\alpha} \geq \frac{\mathbb{P}_a(T > n+1)}{c/(n+1)^\alpha} \cdot \left(\frac{n}{n+1}\right)^\alpha$$

Now, notice $n-1$ and $n+1$ are even, so if we let n go to infinity, both upper and lower bound above will converge to 1. □

Problem 3

Let X, Y be independent standard normal $(0,1)$ random variables.

(a)

Find a for which $U = X + 2Y$, $V = aX + Y$ are independent.

Solution. Note that $U = (1,2) \cdot (X,Y)^T$, $V = (a,1) \cdot (X,Y)^T$, and $(X,Y)^T \sim \mathcal{N}(0, I)$. (U,V) are normal vector, so U, V are independent if and only if $Cov(U,V) = 0$.

$$\begin{aligned}
Cov(U,V) &= (1,2) \cdot I \cdot (a,1)^T \\
&= a + 2 \\
a &= -2
\end{aligned}$$

(b)

Find $\mathbb{E}(XY|X+2Y=a)$ for all $a \in \mathbb{R}$. HINT: Use(a).

Solution. Note that $X = \frac{U-2V}{5}$ and $Y = \frac{2U+V}{5}$. So the expectation turns into:

$$\frac{1}{25}\mathbb{E}(2U^2 - 3UV - 2V^2|U=a) = \frac{1}{25}(2a^2) - 3a \cdot \mathbb{E}(V) - 2 \cdot \mathbb{E}(V^2)) = \frac{2a^2 - 10}{25}$$