MATH 505a Fall 2021 Qual Solution Attempts

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Problem 1

(a)

Let X be a non-negative random variable with finite expectation. Show that

$$\sum_{i=1}^{\infty} \mathbb{P}(X \geq i) \leq E[X] < 1 + \sum_{i=1}^{\infty} \mathbb{P}(X \geq i).$$

Proof. Since X is non-negative,

$$\begin{aligned}\mathbb{E}[X] &= \int_{[0,\infty)} x f(x) dx \\ &= \sum_{i=0}^{\infty} \int_{[i,i+1)} x f(x) dx\end{aligned}$$

Then notice that,

$$i \int_{[i,i+1)} f(x) dx \leq \int_{[i,i+1)} x f(x) dx \leq (i+1) \int_{[i,i+1)} f(x) dx$$

That is,

$$i\mathbb{P}(i \leq X \leq i+1) \leq \int_{[i,i+1)} x f(x) dx \leq (i+1)\mathbb{P}(i \leq X \leq i+1)$$

Plugging into the sum, the lower bound becomes:

$$\begin{aligned}&\sum_{i=0}^{\infty} i\mathbb{P}(i \leq X \leq i+1) \\ =&\sum_{i=1}^{\infty} \mathbb{P}(X \geq i)\end{aligned}$$

Similarly, the upper bound:

$$\begin{aligned}
&\sum_{i=0}^{\infty}(i+1)\mathbb{P}(i \leq X \leq i+1) \\
=&\sum_{i=0}^{\infty} i\mathbb{P}(i \leq X \leq i+1) + \sum_{i=0}^{\infty}\mathbb{P}(i \leq X \leq i+1) \\
=&\sum_{i=1}^{\infty}\mathbb{P}(X \geq i) + 1
\end{aligned}$$

□

(b)

Show that if X takes values only in $\{0, 1, \cdots, n\}$ for some n, then the first inequality in (a) is an equality:

$$\sum_{i=1}^{\infty}\mathbb{P}(X \geq i) = \mathbb{E}[X].$$

Proof. Note that if X only take natural number values, we have $\mathbb{P}(X \geq i) = \sum_{k=i}^{\infty}\mathbb{P}(X = k)$. Plug this into the left hand side:

$$\begin{aligned}
\sum_{i=1}^{\infty}\mathbb{P}(X \geq i) &= \sum_{i=1}^{\infty}\sum_{k=i}^{\infty}\mathbb{P}(X = k) \\
&= \mathbb{P}(X = 1) + \mathbb{P}(X = 2) + \mathbb{P}(X = 3) + \cdots \\
&\qquad + \mathbb{P}(X = 2) + \mathbb{P}(X = 3) + \cdots \\
&\qquad\qquad + \mathbb{P}(X = 3) + \cdots \\
&\qquad\qquad + \cdots \\
&= \sum_{i=1}^{\infty} i\mathbb{P}(X = i) \\
&= \mathbb{E}[X]
\end{aligned}$$

□

(c)

Let M be the minimum value seen in 4 die rolls. Find $\mathbb{E}[M]$. You don't need to simplify to one number, just get an expression in terms of numbers only.

Proof. Note that M only takes values in $\{1,2,3,4,5,6\}$, let X_i denotes the value of i-th dice roll we can use the conclusion from (b) that:

$$\begin{aligned}\mathbb{E}[M] &= \sum_{i=1}^{6} \mathbb{P}(M \geq i) \\ &= \sum_{i=1}^{6} \prod_{j=1}^{4} \mathbb{P}(X_j \geq i) \\ &= \sum_{i=1}^{6} \left(1 - \frac{i-1}{6}\right)^4\end{aligned}$$

□

Problem 2

Suppose X and Y are independent continuous random variables with uniform distribution on $[0,1]$.

(a)

Find the density function of $X+2Y$.

Solution. By conditioning on X, we have:
Case 1. $z \in [0,1)$,

$$\begin{aligned}\mathbb{P}(X+2Y \leq z) &= \int_0^z \frac{1}{2}(z-x)\ dx \\ &= \frac{z^2}{4}\end{aligned}$$

Case 2. $z \in [1,2)$,

$$\begin{aligned}\mathbb{P}(X+2Y \leq z) &= \int_0^1 \frac{1}{2}(z-x)\ dx \\ &= \frac{2z-1}{4}\end{aligned}$$

Case 3. $z \in [2,3]$,

$$\begin{aligned}\mathbb{P}(X+2Y \leq z) &= (z-2) + \int_{z-2}^1 \frac{1}{2}(z-x)\ dx \\ &= z - 2 - \frac{z^2-2z-3}{4}\end{aligned}$$

So compute the pdf by differentiating:

$$f_{X+Y}(z) = \begin{cases} \frac{z}{2} & 0 \leq z < 1 \\ \frac{1}{2} & 1 \leq z < 2 \\ -\frac{z}{2} + \frac{3}{2} & 2 \leq z \leq 3 \end{cases}$$

(b)

Find the joint density function for $X - Y$, $X + Y$.

Solution. Let $U = X + Y$, $V = X - Y$. Then $X = \frac{U+V}{2}$, $Y = \frac{U-V}{2}$. We can compute the absolute value of Jacobian of the map $(u, v) \mapsto (x, y)$:

$$J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = \frac{1}{2}$$

then We have the following:

$$\begin{aligned} f_{U,V}(u, v) &= f_{X,Y}(\frac{u+v}{2}, \frac{u-v}{2}) \cdot |J| \\ &= \frac{1}{2} \mathbb{1}_{0 \le \frac{u+v}{2} \le 1, 0 \le \frac{u-v}{2} \le 1}(u, v) \\ &= \frac{1}{2} \mathbb{1}_{0 \le u+v \le 2, 0 \le u-v \le 2}(u, v) \end{aligned}$$

Problem 3

Consider Bernoulli trials with success probability $p \in (0, 1)$. Let p_n be the probability of an odd number of successes in n trials.

(a)

Express p_n in terms of p_{n-1}.

$$p_n = p \cdot (1 - p_{n-1}) + (1 - p) \cdot p_{n-1}$$

(b)

Based on (a), for what value λ does $p_{n-1} = \lambda$ imply $p_n = \lambda$?

$$\begin{aligned} \lambda &= p \cdot (1 - \lambda) + (1 - p) \cdot \lambda \\ \implies \lambda &= \frac{1}{2} \end{aligned}$$

(c)

Show that $lim_n p_n = \lambda$, the value you found in (b). HINT: Write p_n as $\lambda + \epsilon_n$, for the λ you found in (b).

$$\begin{aligned} \lambda + \epsilon_n &= p \cdot (1 - (\lambda + \epsilon_{n-1})) + (1 - p) \cdot (\lambda + \epsilon_{n-1}) \\ \epsilon_n &= (1 - 2p)\epsilon_{n-1} \\ \overset{(*)}{\implies} \lim_{n \to \infty} \epsilon_n &= \lim_{n \to \infty} (1 - 2p)^{n-1} \epsilon_1 = 0 \end{aligned}$$

$(*) : |1 - 2p| < 1$