MATH 505a Fall 2020 Qual Solution Attempts

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Problem 1

Let X_n have binomial $B(n,p)$ distribution.

(a)

Find $\mathbb{E}(\frac{1}{X_n+1})$. Simplify your answer so it does not involve a sum to n, $n+1$, etc.

Solution.

$$\begin{aligned}
\mathbb{E}\left(\frac{1}{X_n+1}\right) &= \sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k} p^k (1-p)^{n-k} \\
&= \sum_{k=0}^{n} \frac{n!}{(n-k)!(k+1)!} p^k (1-p)^{n-k} \\
&= \sum_{k=0}^{n} \frac{(n+1)!}{(n-k)!(k+1)!} \frac{1}{n+1} p^{k+1} (1-p)^{n-k} \frac{1}{p} \\
&= \frac{1}{(n+1)p} \sum_{k=1}^{n+1} \binom{n+1}{k} p^k (1-p)^{n+1-k} \\
&= \frac{1}{(n+1)p} \left(1-(1-p)^{n+1}\right)
\end{aligned}$$

(b)

Suppose $p = p_n$ and $np_n \to \lambda$ as $n \to \infty$, with $\lambda \in (0,\infty)$. Find $\lim_n \mathbb{E}(\frac{1}{X_n+1})$. Is it the same as $\lim_n \frac{1}{\mathbb{E}(X_n+1)}$?

Solution.

$$\lim_n \mathbb{E}\left(\frac{1}{X_n+1}\right) = \lim_n \frac{1-(1-p)^{n+1}}{(n+1)p}$$
$$= \frac{1}{\lambda}\left(1 - \lim_n \left(1-\frac{np}{n}\right)^{n+1}\right)$$
$$= \frac{1-e^{-\lambda}}{\lambda}$$

It is not same as $\lim_n \frac{1}{\mathbb{E}(X_n+1)} = \frac{1}{\lambda+1}$.

Problem 2

Let X, Y be independent with $X \sim Poisson(\lambda)$ and $Y \sim Poisson(\mu)$ distribution.

(a)

Find $\mathbb{P}(X = k | X + Y = n)$ for $0 \leq k \leq n$. Simplify your answer so it does not involve a sum. Do the actual calculation, don't just cite a theorem.

Solution.

$$\mathbb{P}(X = k|X+Y=n) = \frac{\mathbb{P}(X=k, Y=n-k)}{\sum_{l=0}^{n} \mathbb{P}(X=l, Y=n-l)}$$
$$= \frac{e^{-\lambda}\frac{\lambda^k}{k!}e^{-\mu}\frac{\mu^{n-k}}{(n-k)!}}{\sum_{l=0}^{n} e^{-\lambda}\frac{\lambda^l}{l!}e^{-\mu}\frac{\mu^{n-l}}{(n-l)!}}$$
$$= \frac{\binom{n}{k}\left(\frac{\lambda}{\lambda+\mu}\right)^k\left(\frac{\mu}{\lambda+\mu}\right)^{n-k}}{\sum_{l=0}^{n}\binom{n}{l}\left(\frac{\lambda}{\lambda+\mu}\right)^l\left(\frac{\mu}{\lambda+\mu}\right)^{n-l}}$$
$$= \binom{n}{k}\left(\frac{\lambda}{\lambda+\mu}\right)^k\left(\frac{\mu}{\lambda+\mu}\right)^{n-k}$$

(b)

Find $\mathbb{E}(X^2 + Y^2 | X + Y = n)$.

Solution.

$$\mathbb{E}(X^2+Y^2|X+Y=n) = \sum_{k=0}^{n}(k^2+(n-k)^2)\binom{n}{k}\left(\frac{\lambda}{\lambda+\mu}\right)^k\left(\frac{\mu}{\lambda+\mu}\right)^{n-k}$$
$$= \mathbb{E}(M^2) + \mathbb{E}((n-M)^2)$$

where $M \sim Binomial(n, \frac{\lambda}{\lambda+\mu})$. Given that $\mathbb{E}(M) = \frac{n\lambda}{\lambda+\mu}$ and $Var(M) = \frac{n\lambda\mu}{(\lambda+\mu)^2}$, we have:

$$\mathbb{E}(X^2+Y^2|X+Y=n) = 2\left(\frac{n\lambda\mu}{(\lambda+\mu)^2} + \left(\frac{n\lambda}{\lambda+\mu}\right)^2\right) + n^2 - \frac{2\lambda n^2}{\lambda+\mu}$$

Problem 3

The county hospital is located at the center of a square whose sides are 2 miles wide. If an accident occurs within this square, then the hospital sends out an ambulance. The road network is rectangular, so the travel distance from the hospital, at $(0,0)$, to the point (x,y) is $|x|+|y|$. If an accident occurs at a point that is uniformly distributed in the square, find the mean and variance of the travel distance of the ambulance.

Solution.

$$\begin{aligned}\mathbb{E}(|X|+|Y|) &= \int_{-1}^{1}\int_{-1}^{1}(|x|+|y|)\cdot\frac{1}{4}dxdy\\ &\overset{(*)}{=} \int_{0}^{1}\int_{0}^{1}(x+y)dxdy\\ &= 1\end{aligned}$$

$(*)$ by symmetry.

$$\begin{aligned}Var(|X|+|Y|) &= \mathbb{E}((|X|+|Y|)^2) - (\mathbb{E}(|X|+|Y|))^2\\ &= \int_0^1\int_0^1 (x+y)^2 dxdy - 1\\ &= \frac{7}{6} - 1\\ &= \frac{1}{6}\end{aligned}$$

Problem 4

Let X be a finite set X, and let P and Q be probabilities on X. Define the total variation distance between P and Q by

$$||P-Q||_{TV} = \frac{1}{2}\sum_{x\in X}|P(x)-Q(x)|.$$

Prove that

$$||P-Q||_{TV} = \max_{A\subset X}|P(A)-Q(A)|,$$

where the maximum is over subsets A of X.

Proof. Let $S = \{x\in X : P(X)\geq Q(x)\}$, then,

$$\begin{aligned}||P-Q||_{TV} &= \frac{1}{2}\left(\sum_{x\in S}(P(x)-Q(x)) + \sum_{x\in S^c}(Q(x)-P(x))\right)\\ &= \frac{1}{2}\left(P(S)-Q(S)+Q(S^c)-P(S^c)\right)\\ &\overset{(*)}{=} P(S)-Q(S)\end{aligned}$$

$(*)$ for any $A \subset X$,

$$P(A) + P(A^c) = Q(A) + Q(A^c) = 1 \implies P(A) - Q(A) = Q(A^c) - P(A^c)$$

Now it suffices to show that $\max_{A \subset X} |P(A) - Q(A)| = P(S) - Q(S)$. Given $A \subset X$,

$$\begin{aligned}
|P(A) - Q(A)| &= |(P(A \cap S) + P(A \cap S^c)) - (Q(A \cap S) + Q(A \cap S^c))| \\
&= |(P(A \cap S) - Q(A \cap S)) - (Q(A \cap S^c) - P(A \cap S^c))| \\
&\overset{(**)}{\leq} \max\{P(A \cap S) - Q(A \cap S), Q(A \cap S^c) - P(A \cap S^c)\} \\
&\overset{(***)}{\leq} \max\{P(S) - Q(S), Q(S^c) - P(S^c)\} \\
&\overset{(*)}{=} P(S) - Q(S)
\end{aligned}$$

$(**)$ $P(A \cap S) - Q(A \cap S) \geq 0$, $Q(A \cap S^c) - P(A \cap S^c) \geq 0$ by the definition of S.
$(* * *)$ Any subset $B \subset S$, $0 \leq P(B) - Q(B) \leq P(S) - Q(S)$, by the definition of S. Similarly, any $C \subset S^c$, $0 \leq Q(C) - P(C) \leq Q(S^c) - P(S^c)$. $\square$