MATH 505a Spring 2019 Qual Solution Attempts

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Problem 1

Suppose that each of 5 jobs is assigned at random to one of three servers A,B and C. [For example, one possible outcome would be that job 1 goes to server B, job 2 goes to server C, job 3 goes to server C, job 4 goes to server B and job 5 goes to server A. "At random" here means that there are 3^5 equally likely outcomes]

(a)

Find the probability that server C gets all 5 jobs.

$$\mathbb{P}(\text{C gets 5 jobs}) = \left(\frac{1}{3}\right)^5$$

(b)

Let S be the number of servers that get exactly one job. Find $\mathbb{E}S$.

Solution. Let I_A, I_B, I_C denote the indicator functions of A, B, C get exactly one job, respectively. Then,

$$\begin{aligned}
\mathbb{E}(S) &= \mathbb{E}(I_A + I_B + I_C) \\
&= \mathbb{P}(I_A = 1) + \mathbb{P}(I_B = 1) + \mathbb{P}(I_C = 1) \\
&= 3\binom{5}{1}\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^4 \\
&= \frac{80}{81}
\end{aligned}$$

(c)

Find the probability that no server gets more than 2 jobs.

$$\begin{aligned}
\mathbb{P}(\text{no server gets more than 2 jobs}) &= \mathbb{P}(\text{2 servers get 2 jobs each, 1 server get 1 job}) \\
&= \binom{3}{1}\left(\frac{1}{3}\right)^2\left(\frac{1}{3}\right)^2\left(\frac{1}{3}\right) \\
&= 3\left(\frac{1}{3}\right)^5 \\
&= \frac{1}{81}
\end{aligned}$$

(d)

Take the same story, but with m in pace of 5 for the number of jobs, and n in place of 3 for the number of servers. Find the variance of S, in terms of m and n.

Solution. Let I_i denotes the indicator function that the i-th server gets exactly 1 job.

$$\begin{aligned}
Var(S) &= \mathbb{E}(S^2) - (\mathbb{E}S)^2 \\
&= \mathbb{E}(\sum_{i=1}^{n} I_i^2 + 2\sum_{i\neq j}^{n} I_i \cdot I_j) - \left(\sum_{i=1}^{n}\mathbb{E}(I_i)\right)^2 \\
&= \sum_{i=1}^{n}\mathbb{P}(I_i = 1) + 2\sum_{i\neq j}^{n}\mathbb{P}(I_i = 1, I_j = 1) - \left(\sum_{i=1}^{n}\mathbb{P}(I_i = 1)\right)^2 \\
&= m\left(\frac{n-1}{n}\right)^{m-1} - \left(m\left(\frac{n-1}{n}\right)^{m-1}\right)^2 + 2\cdot n(n-1)\cdot m(m-1)\cdot\left(\frac{1}{n}\right)^2\left(\frac{n-2}{n}\right)^{m-2}
\end{aligned}$$

Problem 2

(a)

Suppose that X is Poisson with parameter λ. Find the characteristic function of X.

$$\begin{aligned}
\mathbb{E}(e^{itX}) &= \sum_{k=0}^{\infty} e^{itk}e^{-\lambda}\frac{\lambda^k}{k!} \\
&= e^{-\lambda(1-e^{it})}\sum_{k=0}^{\infty} e^{-e^{it}\lambda}\frac{(e^{it}\lambda)^k}{k!} \\
&= e^{-\lambda(1-e^{it})}
\end{aligned}$$

(b)

Suppose that X_n is Poisson with parameter λ_n and that $\lambda_n \to \infty$. Show using characteristic functions that $(X_n - \lambda_n)/\sqrt{\lambda}$ converges in distribution, and describe the limiting distribution.

Proof.

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{E}(e^{it(X_n-\lambda_n)/\sqrt{\lambda}}) &= \lim_{n\to\infty} \exp\left(-it\sqrt{\lambda_n} - \lambda_n + \lambda_n e^{i\frac{t}{\sqrt{\lambda_n}}}\right) \\
&= \lim_{n\to\infty} \exp\left(-it\sqrt{\lambda_n} - \lambda_n + \lambda_n \sum_{k=0}^{\infty} \frac{(it/\sqrt{\lambda_n})^k}{k!}\right) \\
&= \lim_{n\to\infty} \exp\left(-it\sqrt{\lambda_n} + \lambda_n(it/\sqrt{\lambda_n}) + \lambda_n \frac{(it/\sqrt{\lambda_n})^2}{2} + \lambda_n \frac{(it/\sqrt{\lambda_n})^3}{6} + \cdots\right) \\
&= \lim_{n\to\infty} \exp\left(-\frac{t^2}{2} - \frac{it^3}{6\lambda^{1/2}} + \cdots\right) \\
&= \exp\left(-\frac{t^2}{2}\right)
\end{aligned}$$

It converges to the standard normal distribution. □

Problem 3

A stick of length 1 is broken into two pieces at a uniformly distributed random point.

(a)

Find the expected length of the smaller piece.

Solution. Let X_1, X_2, U_1 denote the length of the smaller stick, the length of the larger stick, and the location of the first break point, respectively.

$$\begin{aligned}
F_{X_1}(x) &= \mathbb{P}(X_1 \le x, U_1 \le 1/2) + \mathbb{P}(X_1 \le x, U_1 > 1/2) \\
&= x + (1 - (1 - x)) \\
&= 2x
\end{aligned}$$

$$f_{X_1}(x) = 2,\ 0 < x < 1/2$$

Similarly, we have

$$F_{X_2}(x) = 1 - \mathbb{P}(X_1 < 1 - x) = 2x - 1$$

$$f_{X_2}(x) = 2,\ 1/2 < x < 1$$

From the pdf of X_1, we have,

$$\mathbb{E}(X_1) = \frac{1}{4}$$

(b)

Find the expected value of the ratio of the smaller length over the larger.

Solution.

$$\begin{aligned}\mathbb{P}(\frac{X_1}{X_2} \leq t) &= \mathbb{P}(\frac{X_1}{1-X_1} \leq t) \\ &= \mathbb{P}(X_1 \leq \frac{t}{t+1}) \\ &= \frac{2t}{t+1}, \ 0 < t < 1\end{aligned}$$

Since the ratio only takes non-negative value, we can use complementary cdf to compute expectation:

$$\begin{aligned}\mathbb{E}\left(\frac{X_1}{X_2}\right) &= \int_0^1 1 - \frac{2t}{1+t} dt \\ &= 1 - 2 + 2\int_0^1 \frac{1}{1+t} dt \\ &= -1 + 2\ln(2)\end{aligned}$$

(c)

Suppose the larger piece is then broken at a random point, uniformly distributed over its length, independent of the first break point. There are then three pieces. Find the probability the longest of the three has length more than 1/2.

Solution. Let X_3, U_2 be the length of the larger piece of the previous larger piece and the second break point location (start from the left end of the X_2). First notice that the pdf of U_2:

$$F_{U_2|X_2=x}(t) = \frac{t}{x}$$

Since $\mathbb{P}(X_1 > 1/2) = 0$, so the desired probability is

$$\begin{aligned}\mathbb{P}(X_3 > 1/2) &= \int_{1/2}^1 \mathbb{P}(X_3 > 1/2, U_2 > x/2) + \mathbb{P}(X_3 > 1/2, U_2 \leq x/2) \cdot f_{X_2}(x) dx \\ &= \int_{1/2}^1 [\mathbb{P}(U_2 > 1/2|X_2 = x) + \mathbb{P}(U_2 < X_2 - 1/2|X_2 = x) \cdot 2dx \\ &= 2\int_{1/2}^1 1 - \frac{1}{2x} + \frac{1}{x}(x - \frac{1}{2}) dx \\ &= 2 + 2\ln(1/2)\end{aligned}$$