MATH 505a Fall 2019 Qual Solution Attempts

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Problem 1

Suppose A, B, C are pairwise independent, $A \cap B \cap C = \emptyset$, and $\mathbb{P}(A) = \mathbb{P}(B) = \mathbb{P}(C) = p$.

(a)

What is the largest possible value of p?

Solution. First, notice that $A \cap B \cap C = \emptyset \implies (A \cap B) \cap (B \cap C) = \emptyset$. Then

$$p = \mathbb{P}(B) \geq \mathbb{P}(A \cap B) + \mathbb{P}(B \cap C) = \mathbb{P}(A)\mathbb{P}(B) + \mathbb{P}(B)\mathbb{P}(C) = 2p^2$$

We have

$$p \leq \frac{1}{2}$$

Then, we let $\mathbb{P}(A) = \mathbb{P}(B) = \frac{1}{2}$. ($\mathbb{P}(A \cap B) = \frac{1}{4}$ and $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) = \frac{3}{4}$). And let $C = (A \cup B) \backslash (A \cap B)$. Now we want to show that such A, B, C satisfies the assumptions.

$$\mathbb{P}(C) = \mathbb{P}(A \cup B) - \mathbb{P}(A \cap B) = \frac{1}{2}$$

$$\mathbb{P}(C \cap A) = \mathbb{P}(A) - \mathbb{P}(A \cap B) = \frac{1}{4}$$

$$\mathbb{P}(C \cap B) = \mathbb{P}(B) - \mathbb{P}(A \cap B) = \frac{1}{4}$$

$$A \cap B \cap C = (A \cap B) \cap ((A \cup B) \backslash (A \cap B)) = \emptyset$$

Therefore $\frac{1}{2}$ is the largest possible value for p.

(b)

Is it possible that $\mathbb{P}(A \cup B \cup C) = 1$? Prove or disprove.

Solution. Impossible.

Proof. By inclusion-exclusion theorem,

$$\begin{aligned}\mathbb{P}(A \cup B \cup C) &= \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - (\mathbb{P}(A \cap B) + \mathbb{P}(A \cap C) + \mathbb{P}(B \cap C)) + \mathbb{P}(A \cap B \cap C) \\ &= 3p - 3p^2\end{aligned}$$

which reaches maximum at $p = 0.5$ and $\mathbb{P}(A \cup B \cup C) = 0.75$. □

Problem 2

Consider two coins: coin 1 shows heads with probability p_1 and coin 2 shows heads with probability p_2. Each coin is tossed repeatedly. Let T_i be the time of first heads for coin i, and define the event $A = \{T_1 < T_2\}$.

(a)

Find $\mathbb{P}(A)$. HINT: One possible method is to condition on one of the variables.

Solution. Notice that $T_i \sim \text{Geometric}(p_i)$.

$$\begin{aligned}\mathbb{P}(A) &= \sum_{t=2}^{\infty} \mathbb{P}(T_1 \leq t-1)\mathbb{P}(T_2 = t) \\ &= \sum_{t=2}^{\infty} (1 - (1-p_1)^{t-1})(p_2(1-p_2)^{t-1}) \\ &= p_2 \left(\sum_{t=1}^{\infty} (1-p_2)^t - \sum_{t=1}^{\infty} [(1-p_2)(1-p_1)]^t \right) \\ &= \frac{p_1(1-p_2)}{1-(1-p_1)(1-p_2)}\end{aligned}$$

(b)

Find $\mathbb{P}(T_1 = k|A)$ for all $k \geq 1$.

Solution.

$$\begin{aligned}\mathbb{P}(T_1 = k|A) &= \frac{\mathbb{P}(T_1 = k)\mathbb{P}(T_2 > k)}{\mathbb{P}(A)} \\ &= \frac{[1-(1-p_1)(1-p_2)](1-p_1)^{k-1}p_1(1-p_2)^k}{p_1(1-p_2)} \\ &= [1-(1-p_1)(1-p_2)](1-p_1)^{k-1}(1-p_2)^{k-1}\end{aligned}$$

Problem 3

Player A and B are having a table tennis match; the first player to win 3 games wins the match. One of the players is better than the other; this better player wins each game with probability 0.7. Carl comes to watch the match. He does not know who is the better player so (based on Carl's information) A, B each initially have probability 0.5 to be the better player. Then Carl sees A win 2 of the first 3 games.

(a)

What is now the probability (after the 3 games, based on Carl's information) that A is the better player? Simplify your answer to a single fraction or decimal.

Solution. Let $C = \{\text{A won 2 of the first 3 games}\}$, $A = \{\text{A is better}\}$, and $B = \{\text{B is better}\}$.

$$\begin{aligned}
\mathbb{P}(A|C) &= \mathbb{P}(C|A)\frac{\mathbb{P}(A)}{\mathbb{P}(C)} \\
&= \binom{3}{2}(0.7)^2(0.3)\frac{0.5}{\mathbb{P}(C\cap A)+\mathbb{P}(C\cap B)} \\
&= \binom{3}{2}(0.7)^2(0.3)\frac{0.5}{\binom{3}{2}(0.7)^2(0.3)0.5+\binom{3}{2}(0.3)^2(0.7)0.5} \\
&= \frac{7}{10}
\end{aligned}$$

(b)

What is now the probability (after 3 games, based on Carl's information) that A will go on to win the match? NOTE: Express your answer for (b) in terms of numbers; you do not need to simplify to a single number. Ananswer in a form like $\frac{5}{4} + 7(2 - \frac{9}{5})$ is OK.

Solution. Let $W_A = \mathbb{P}(\text{A win a game}|C) = 0.7\cdot 0.7 + 0.3\cdot 0.3 = 0.58$, so $W_B = \mathbb{P}(\text{B win a game}|C) = 0.42$. So,

$$\begin{aligned}
\mathbb{P}(\text{A win the match}|C) &= W_A + W_B W_A \\
&= 0.58 + 0.42\cdot 0.58 \\
&= 0.8236
\end{aligned}$$

Problem 4

Suppose X_n is binomial with parameters (n, p) with $0 \le p \le 1$, and X is Poisson(λ).

(a)

Find the moment generating function of X_n.

Solution.

$$\begin{aligned}\mathbb{E}(e^{X_n t}) &= \sum_{k=0}^{n} e^{tk}\binom{n}{k}p^k(1-p)^{n-k} \\ &= \sum_{k=0}^{n}\binom{n}{k}p(pe^t)^k(1-p)^{n-k} \\ &\overset{(*)}{=} (1-p+pe^t)^n\end{aligned}$$

$(*)$ Binomial formula.

(b)

Suppose $n \to \infty$ and $p = p_n \to 0$ with $np \to \lambda \in (0,\infty)$. Show that $\mathbb{P}(X_n = k) \to \mathbb{P}(X = k)$ as $n \to \infty$, for all $k \geq 0$. HINT:$(1-\frac{c_n}{n})^n \to e^{-c}$ if $c_n \to c$.

Proof.

$$\begin{aligned}\lim_{n\to\infty}\mathbb{P}(X_n = k) &= \lim_{n\to\infty}\binom{n}{k}p^k(1-p)^{n-k} \\ &= \lim_{n\to\infty}\frac{1}{k!}\frac{n!}{(n-k)!}p^k(1-\frac{np}{n})^n(1-\frac{np}{n})^{-k} \\ &= \frac{1}{k!}\lim_{n\to\infty}\frac{n-k+1}{n}\frac{n-k+2}{n}\cdots\frac{n-1}{n}\frac{n}{n}(np)^k(1-\frac{np}{n})^n(1-\frac{np}{n})^{-k} \\ &= \frac{1}{k!}\cdot 1\cdot 1\cdots 1\cdot 1\cdot \lambda^k\cdot e^{-\lambda}\cdot 1 \\ &= e^{-\lambda}\frac{\lambda^k}{k!}\end{aligned}$$

□

(c)

For n, p as in part(b), show that $\mathbb{P}(X_n > k) \to \mathbb{P}(X > k)$ as $n \to \infty$, for all $k \geq 0$.

Proof.

$$\begin{aligned}\lim_{n\to\infty}\mathbb{P}(X_n > k) &= \lim_{n\to\infty}(1-\mathbb{P}(X_n \leq k) \\ &= 1 - \lim_{n\to\infty}\sum_{x=0}^{k}\mathbb{P}(X_n = x) \\ &= 1 - \sum_{x=0}^{k}\lim_{n\to\infty}\mathbb{P}(X_n = x) \\ &\overset{(**)}{=} 1 - \sum_{x=0}^{k}\mathbb{P}(X = x) \\ &= \mathbb{P}(X > k)\end{aligned}$$

$(**)$ Conclusion of (b). □