MATH 505a Spring 2018 Qual Solution Attempts

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Problem 1

Let X and Y be independent standard normal random variables and define $V = \min(X, Y)$. Compute the probability density function of V^2. The final answer should be an elementary function.

Solution. Let ϕ denote the cdf of standard normal, and by the symmetry of the standard normal distribution:

$$\begin{aligned}\mathbb{P}(V \leq t) &= 1 - \mathbb{P}(V > t) \\ &= 1 - \mathbb{P}(X > t)\mathbb{P}(Y > t) \\ &\overset{(*)}{=} 1 - \phi(-t)^2\end{aligned}$$

For $V^2,\ t > 0$:

$$\begin{aligned}\mathbb{P}(V^2 \leq t^2) &= \mathbb{P}(-t \leq V \leq t) \\ &= \mathbb{P}(V \leq t) - \mathbb{P}(V \leq -t) \\ &= \phi(t)^2 - \phi(-t)^2\end{aligned}$$

By differentiate, we have:

$$\begin{aligned}f_{V^2}(x) &= \frac{d}{dx}(\phi(\sqrt{x})^2 - \phi(-\sqrt{x})^2) \\ &= 2\phi(\sqrt{x})f_X(\sqrt{x})\frac{1}{2\sqrt{x}} - 2\phi(-\sqrt{x})f_X(-\sqrt{x})\frac{-1}{2\sqrt{x}} \\ &= \frac{1}{\sqrt{2\pi x}}e^{-x/2}(\phi(\sqrt{x}) + \phi(-\sqrt{x})) \\ &\overset{(*)}{=} \frac{1}{\sqrt{2\pi x}}e^{-x/2}\end{aligned}$$

$(*)$ Symmetry of the standard normal.

Problem 2

Consider positions 1 to n arranged in a circle, so that 2 comes after 1, 3 comes after 2, ..., n comes after $n-1$, and 1 comes after n. Similarly, take 1 to n as values, with cyclic order, and consider all $n!$ ways to assign values to positions, bijectively, with all $n!$ possibilities equally likely. For $i = 1$ to n, let X_i be the indicator that position i and the one following are filled in with two consecutive values in increasing order, and define

$$S_n = \sum_{i=1}^{n} X_i, \quad T_n = \sum_{i=1}^{n} iX_i$$

For example, with $n = 6$ and the circular arrangement 314562, we get $X - 3 = 1$ since 45 are consecutive in increasing order, and similarly $X_4 = X_6 = 1$, so that $S_6 = 3$, $T_6 = 13$.

(a)

Compute the mean and the variance of S_n.

Solution.

$$\begin{aligned}
\mathbb{E}(S_n) &= \sum_{i=1}^{n} \mathbb{E}(X_i) \\
&= \sum_{i=1}^{n} \mathbb{P}(X_i = 1) \\
&= n \cdot \frac{n-1}{n(n-1)} \\
&= 1 \\
\mathbb{E}(X_i^2) &= \mathbb{E}(X_i) = 1 \\
\mathbb{E}(X_i X_j) &= \begin{cases} \frac{n-2}{n(n-1)(n-2)}, & |i-j| = 1 \\ \frac{n-2}{n(n-1)(n-2)}, & |i-j| > 1 \end{cases}
\end{aligned}$$

$$\begin{aligned}
Var(S_n) &= \mathbb{E}(S_n^2) - \mathbb{E}(S_n)^2 \\
&= \sum_{i=1}^{n} \mathbb{E}(X_i^2) + \sum_{i \neq j}^{n} \mathbb{E}(X_i X_j) - \mathbb{E}(S_n)^2 \\
&= 1 + n(n-1)\frac{n-2}{n(n-1)(n-2)} - 1 \\
&= 1
\end{aligned}$$

(b)

Compute the mean and variance of T_n.

Solution.

$$\begin{aligned}
\mathbb{E}(T_n) &= \sum_{i=1}^{n} i\mathbb{E}(X_i) \\
&= \sum_{i=1}^{n} i \cdot \frac{1}{n} \\
&= \frac{1}{n}\frac{n(n+1)}{2} \\
&= \frac{1+n}{2} \\
\mathbb{E}(T_n^2) &= \mathbb{E}(\sum_{i=1}^{n} X_i)^2 \\
&= \mathbb{E}(\sum_{i,j}^{n} ijX_iX_j) \\
&= \sum_{i,j}^{n} ij\mathbb{E}(X_iX_j) \\
&= \frac{1}{n(n-1)}\sum_{i,j}^{n} ij \\
&= \frac{1}{n(n-1)}(\sum_{i=1}^{n} i)^2 \\
&= \frac{1}{n(n-1)}(\frac{(n+1)n}{2})^2 \\
&= \frac{n(n+1)^2}{4(n-1)} \\
Var(T_n) &= \mathbb{E}(T_n^2) - \mathbb{E}(T_n)^2 \\
&= \frac{n(n+1)^2}{4(n-1)} - \frac{(1+n)^2}{4}
\end{aligned}$$

Problem 3

A box is filled with coins, each giving heads with some probability p. The value of p varies from one coin to another, and it is uniform in [0,1]. A coin is selected at random; that one coin is tossed multiple times. HINT: $\int_0^1 x^m(1-x)^l dx = \frac{m!l!}{(m+l+1)!}$ for nonnegative integers $m,\ l$.

(a)

Compute the probability that the first two tosses are both heads.

Solution.

$$\begin{aligned}\mathbb{P}(\text{head twice}) &= \int_0^1 \mathbb{P}(\text{head twice}|p=t)f_p(t)dt\\ &= \int_0^1 t^2 dt\\ &= \frac{1}{3}\end{aligned}$$

(b)

Let X_n be the number of heads in the first n tosses. Compute $\mathbb{P}(X_n = k)$ for all $0 \leq k \leq n$.

Solution. By the hint,

$$\begin{aligned}\mathbb{P}(X_n) &= \binom{n}{k}\int_0^1 p^k(1-p)^{n-k}dp\\ &= \binom{n}{k}\frac{k!(n-k)!}{(n+1)!}\\ &= \frac{1}{n+1}\end{aligned}$$

(c)

Let N be the number of tosses needed to get heads for the first time. Compute $\mathbb{P}(N = n)$ for all $n \leq 1$.

Solution.

$$\begin{aligned}\mathbb{P}(N=n) &= \int_0^1 (1-p)^{n-1}p\ dp\\ &= \frac{(n-1)!}{(n+1)!}\\ &= \frac{1}{n(n+1)}\end{aligned}$$

(d)

Compute the expected value of N.

Solution.

$$\begin{aligned}\mathbb{E}(N) &= \sum_{n=1}^{\infty}\frac{1}{n+1}\\ &= \infty\end{aligned}$$